Vol. 87, No. 4, October 2014

MAA

MATHEMATICS MAGAZINE

Painting Gabriel's Horn

- Assigning students to first-year seminars
- Mind Switches in *Futurama* and *Stargate*
- Gabriel's Horn, with more dimensions

An Official Publication of the **Mathematical Association of America**

EDITORIAL POLICY

Mathematics Magazine aims to provide lively and appealing mathematical exposition. The *Magazine* is not a research journal, so the terse style appropriate for such a journal (lemma-theorem-proof-corollary) is not appropriate for the *Magazine*. Articles should include examples, applications, historical background, and illustrations, where appropriate. They should be attractive and accessible to undergraduates and would, ideally, be helpful in supplementing undergraduate courses or in stimulating student investigations. Manuscripts on history are especially welcome, as are those showing relationships among various branches of mathematics and between mathematics and other disciplines.

A more detailed statement of author guidelines appears in this *Magazine*, Vol. 83, at pages 73–74, and is available at the *Magazine*'s website www.maa.org/pubs/mathmag.html. Manuscripts to be submitted should not be concurrently submitted to, accepted for publication by, or published by another journal or publisher.

Please submit new manuscripts by email directly to the editor at mathmag@maa.org. A brief message containing contact information and with an attached PDF file is preferred. Word-processor and DVI files can also be considered. Alternatively, manuscripts may be mailed to Mathematics Magazine, 132 Bodine Rd., Berwyn, PA 19312-1027. If possible, please include an email address for further correspondence.

Cover image by Susan Stromquist

MATHEMATICS MAGAZINE (ISSN 0025-570X) is published by the Mathematical Association of America at 1529 Eighteenth Street, N.W., Washington, D.C. 20036 and Lancaster, PA, bimonthly except July/August.
The annual subscription price for *MATHEMATICS MAGAZINE* to an individual member of the Association is $131. Student and unemployed members receive a 66% dues discount; emeritus members receive a 50% discount; and new members receive a 20% dues discount for the first two years of membership.)

Subscription correspondence and notice of change of address should be sent to the Membership/Subscriptions Department, Mathematical Association of America, 1529 Eighteenth Street, N.W., Washington, D.C. 20036. Microfilmed issues may be obtained from University Microfilms International, Serials Bid Coordinator, 300 North Zeeb Road, Ann Arbor, MI 48106.

Advertising correspondence should be addressed to

MAA Advertising
1529 Eighteenth St. NW
Washington DC 20036

Phone: (877) 622-2373
E-mail: tmarmor@maa.org

Further advertising information can be found online at www.maa.org

Change of address, missing issue inquiries, and other subscription correspondence:

MAA Service Center, *maahq@maa.org*

All at the address:

The Mathematical Association of America
1529 Eighteenth Street, N.W.
Washington, DC 20036

Periodicals postage paid at Washington, D.C. and additional mailing offices.

Postmaster: Send address changes to Membership/Subscriptions Department, Mathematical Association of America, 1529 Eighteenth Street, N.W., Washington, D.C. 20036-1385.

Printed in the United States of America

Vol. 87, No. 4, October 2014

MATHEMATICS MAGAZINE

LETTER FROM THE EDITOR

Operations Research—broadly defined—means improving the results of repeated activities by the application of scientific methods and, especially, mathematics. Our first article is a fine example of the art, in which the repeated activity is assigning students to first-year seminars. As many of us know from experience, good assignments can improve peoples' lives. Authors Andrew Beveridge and Stan Wagon describe some of the essential steps. They have worked closely with the "customers" to understand which results to optimize, and have repeatedly tested their final product.

Of course, some applications are more immediately practical than others. You never know when you might encounter a mind-swapping machine like the ones in the article by Ron Evans and Lihua Huang, and need to use their tools for constructing permutations from pairwise exchanges subject to constraints. But even if you never see a mind-switching machine—or if you see one, but you don't want to reverse the results—tools like these are constantly useful in sorting and counting applications.

The article by Vincent Coll and Michael Harrison begins with "Gabriels Horn," a surface of revolution that has infinite surface area but encloses finite volume. When it was first reported, it raised philosophical problems as well as mathematical ones. Now we are mostly at peace with the philosophical questions, but the mathematical aspects can still be extended and considered from other points of view.

Stephan Berendonk's short note gives a simple proof of a property of ellipses. He shows why we need many different proofs of basic results like this. People come to these problems from different backgrounds, with different definitions, and willing to make different assumptions. What seems like a proof to one reader might seem to another reader like missing the point. This is one of the hazards of teaching the foundations, and we should appreciate that when it is well done.

Elsewhere in the Notes Section, Burkard Polster addresses a concern of a surfer dude, and Jorgen Berglund and Ron Taylor have composed some reflections that may help us to realize our identities.

I would mention our Reviews Section, but many of our most experienced readers have probably read it first, before turning to this letter.

We are glad to report the winners of the Allendoerfer prize, for articles that appeared in this MAGAZINE in 2014. The recipients are Sally Cockburn and Joshua Lesperance for "Deranged Socks," and Susan Marshall and Donald Smith for "Feedback, Control, and the Distribution of Prime Numbers."

A Special IMO Feature. At the invitation of Team Captain Po-Shen Loh, each of the members of the USA team has contributed a solution to one of July's IMO problems. This gives us an opportunity to see how these medal winners thought about the problems. As always in October, we also have the problems and solutions from the USAMO and USAMO, and our own evergreen Problems Section, enough to challenge all of us.

Walter Stromquist, Editor

ARTICLES

The Sorting Hat Goes to College

ANDREW BEVERIDGE
Macalester College
St. Paul, MN 55105
abeverid@macalester.edu

STAN WAGON
Macalester College
St. Paul, MN 55105
wagon@macalester.edu

In the Harry Potter stories [**6**], each new year at the Hogwarts School for Witchcraft and Wizardry starts with the ceremonial assignment of new first-year students to one of the four houses: Griffindor, Hufflepuff, Ravenclaw, and Slytherin. This is a milestone for the young students because their assigned houses greatly influence their future directions. The crucial assignment process is entrusted to the magical Sorting Hat, which provides a moment of revelation for the student. Indeed, this sentient garment uses its magical perception to reveal the student's personality and strengths, and consequently places the student into the best-suited house. However, we cannot help but observe another pattern in the assignments. Every year, the 40 students are split evenly, both among houses and by gender. Perhaps the Hat respects some constraints as it divides the students.

This constrained optimization problem captures a fanciful moment of high drama. However, a strikingly similar process takes place annually on college campuses across the United States. Rather than assigning students to houses, colleges must distribute incoming students among an offering of first-year seminars. These courses ease the transition into college by introducing the paradigms, expectations, and standards of higher education [**2**]. Like the houses of Hogwarts, many first-year seminars strive to develop community within the classroom, the student body, and the institution as a whole. Some colleges use seminars with uniform content, but many offer a variety of discipline-specific seminars whose content is as disparate as the interests of the faculty. In this latter setting, assigning first-years to seminars becomes remarkably similar to the job of the Sorting Hat.

Macalester College, a liberal arts college with about 500 incoming students each year, offers discipline-specific first-year courses. Macalester expends great institutional effort in developing these courses and making them effective. Likewise, matching each student to a desired course is taken very seriously. Prior to 2009, this assignment was performed by hand using a labor-intensive process. This process was ripe for improvement, both in terms of efficiency and quality of assignment. Without access to sorcery, we decided to use the next best thing: mathematics! Our first solution (presented in detail in [**3**]) used the classic Minimum Weight Perfect Matching algorithm (also known as the Hungarian algorithm) from graph theory, with excellent results.

More recently, we set out to improve the flexibility of the process. For example, it can happen that the ideal maximum class size of 16 cannot be enforced and that some

Math. Mag. **87** (2014) 243–251. `doi:10.4169/math.mag.87.4.243`.

classes of size 17 are needed. The Hungarian method would have to know in advance which classes are to be allowed to exceed 16. A more general optimization method would not require such specific information, but do what is best to maximize student happiness while simultaneously minimizing the number of 17-sized classes. We were able to develop an optimization method that does exactly this using Integer Linear Programming (ILP). This approach also gives us the ability to handle a wider variety of constraints that do not fit easily into a graph theory context.

In any real-world application, a central task is to state the problem mathematically so that the client's goals are met and the problem is solvable. This almost always calls for certain trade-offs: The graph approach uses a blindingly fast algorithm, but can require manual intervention when the data has certain structural problems. While ILP can fail to solve very large problems, it is fast enough for the 2000-variable instances that arise here and more easily achieves the client's goals. Using ILP for a much larger problem, say 10000 students instead of 500, we would likely have to try several methods and perhaps settle for an approximation to the true optimum.

This paper traces the evolution of the process, with full details of how to set up the optimization problem using ILP and a discussion of the resulting benefits.

First-year course assignment at Macalester

Every department at Macalester offers one or more first-year seminars, called First-Year Courses (FYCs). In a given year, no two FYCs are alike. During early summer, students are given course descriptions and they rank their top four choices. The Academic Programs office (AP) compiles the student preferences and is responsible for assigning students to courses, while respecting some modest constraints on the enrollment profile of each seminar. Typically, about 500 incoming students must be placed into about 35 FYCs. The goal is to place each student into a course that he or she ranked, while also trying to maximize overall student satisfaction. Roughly speaking, this means trying to have many first- and second-choice placements, and far fewer third- and fourth-choice placements.

Academic Programs also tries to enforce the following constraints, designed to keep a uniform experience among classes. First, each class size should be from 10 to 16 students. Second, the demographics of each class should be roughly comparable to the entire student body. Historically, this has meant enforcing a gender balance by reserving 4 seats for men and 4 seats for women. Macalester's student body is 60% female, so this constraint becomes relevant in a few seminars each year. Finally, international students make up 11% of the student body, and AP has found it useful to cap the number of international students in any course at 8.

In a given year, it may not be feasible to assign each student to a ranked choice while enforcing all of the constraints. In these situations, AP allows some isolated constraint violations so that every student ends up in a desired course; the alternative—placing a student in an unranked course—is deemed unacceptable.

Prior to 2009, the assignment was performed by hand. Each student's four preferences were listed on a sheet of paper and these were organized into piles, each pile representing a seminar. Two staff members would spend a week looking for beneficial chains of swaps that would improve the overall profile of the assignment. Ultimately, this method worked well: FIGURE 1 shows some historical results for these manual assignments. On average, AP placed 87% of students into their first or second choices. Below, we will show how to get an assignment for the 2004 data that puts 97.3% into their top two choices (as opposed to the 87.4% from FIGURE 1).

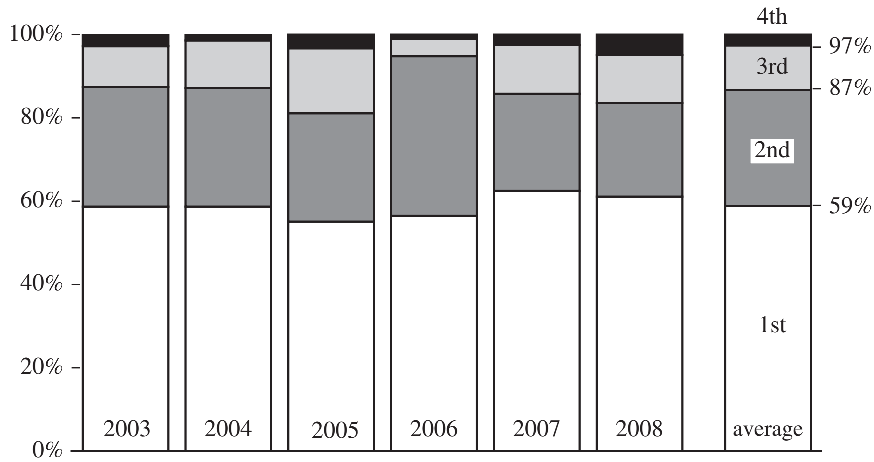


Figure 1 Manual assignment profiles, 2003–2008

The Hungarian Sorting Hat

In 2008, the first author and Macalester senior Sean Cooke tackled the FYC Assignment Problem in a senior seminar in Combinatorial Optimization. Our goal was to automate the process by converting the problem to an Assignment Problem (described below). In addition to the benefits of automation, we were confident that we could also improve the quality of the assignment. We approached this as a consulting project, where the "client" was the AP office. Through a series of meetings with them, we identified their requirements, constraints, and priorities. We used historical data to fine-tune our solution. Once we had a good prototype, we solicited feedback on our assignments for this data from the AP office. Their responses guided further improvements that improved the quality of our assignments. This sustained dialog was crucial to the final design and also to successfully "sell the client" on adopting the algorithm. In particular, we showed AP that our algorithm, dubbed the Hungarian Sorting Hat because of its use of the Hungarian matching algorithm, outperformed their manual assignments on the historical data. They adopted our automated framework in summer 2009, and have used it ever since, but with an algorithm change in 2013.

FIGURE 2 shows the performance of the Hungarian Hat from 2009 to 2012. The average (1st, 2nd, 3rd, 4th) percentages for the manual assignments of FIGURE 1 were 58.8%, 27.9%, 10.7%, 2.6%. The average performance of the Hungarian Hat is 59.5%, 31.3%, 8.3%, 0.9%. Even though the years are different than those in FIGURE 1, this improvement is typical for the comparison: The Hungarian Hat placed 4.1% more students in their first or second choices, compared with the manual assignments, as well as many fewer into a fourth choice.

We now describe the mathematics behind the Hungarian Sorting Hat. The FYC Assignment Problem is a variant of the classic Assignment Problem [**5**]. Suppose that we have n workers and n jobs to be filled. Each worker is qualified to perform a subset of the jobs, and the cost of the assignment depends on the worker and the job. The objective is to find a matching of workers to jobs that minimizes total cost. This situation can be modeled by a weighted bipartite graph, where the n vertices in one part correspond to the workers and the n vertices in the other part represent the jobs. If worker i is qualified to perform job j at cost c_{ij}, then we connect the corresponding vertices with an edge of that weight. This optimization problem is solved by the very fast Hungarian Algorithm, developed by H. W. Kuhn, who was extending the work

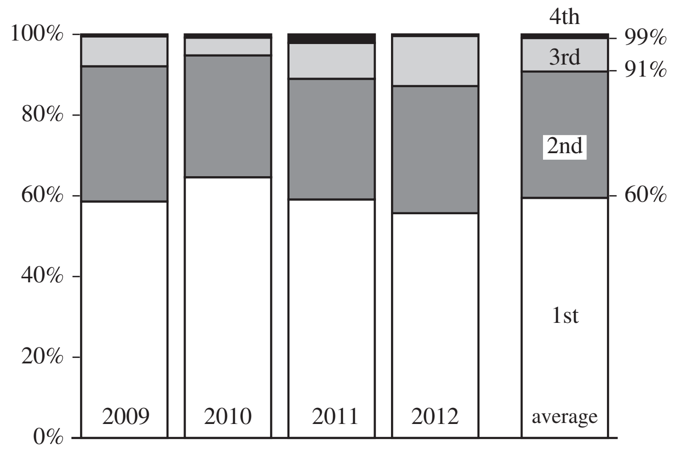


Figure 2 Assignments by the Hungarian Sorting Hat, 2009–2012

of the Hungarian mathematicians Kőnig and Egerváry. A detailed description of the method is in [**4**, ch. 5]. The algorithm analyzes the weighted bipartite graph, and returns a minimum weight perfect matching. This matching corresponds to an optimal assignment of workers to jobs. The basis of the algorithm is the use of swaps along augmenting paths, a very natural idea and one that was used in an informal way by the AP staff when the job was handled manually.

Our main challenge was to convert the FYC Assignment Problem into a classic Assignment Problem. This involved two steps. First, we created a bipartite graph that enforced the college constraints on class size and class demographics. Second, we chose a weighting scheme that ensured that the Hungarian Algorithm made trade-offs that were consistent with the goals and priorities of the AP office.

We summarize the features of the resulting graph. For a more thorough explanation, see [**3**]. In order to run the Hungarian Algorithm, we need a bipartite graph G with parts X and Y, where $|X| = |Y|$, so we start by describing these sets. The set X will represent the students and the set Y will represent the classes. For each class, we place 16 vertices into Y, one for each seat available. Next, we place a vertex in X for each student. Of course, there are more class seats than students, so we add some additional "dummy" vertices to X to restore balance. These dummy vertices will become empty class seats.

Next, we describe the edge structure of G from the seminar perspective; a small part of the graph is in FIGURE 3. This edge structure enforces class size and demographic constraints. A given seminar has 16 corresponding vertices. Eight of these vertices are adjacent to every student interested in the course. Four of the vertices are adjacent only to interested U.S. women, and the final 4 vertices are adjacent exclusively to interested U.S. men. A fully-enrolled class must have at least 4 U.S. women, 4 U.S. men, and a maximum of 8 international students. Finally, each dummy vertex is adjacent to one U.S. woman's seat, one U.S. man's seat, and four generic seats. This means that there can be at most 6 unfilled seats, for a minimum enrollment of 10 students. Furthermore, a class that is below maximum size is only required to have 3 U.S. women and 3 U.S. men.

This brings us to the edge weights. We ultimately settled on a geometric weighting scheme, where the weights of first, second, third, and fourth choice classes were given by the weight vector $\left(1, \alpha, \alpha^2, \alpha^3\right)$, where $\alpha > 1$. The value of α is chosen according to the trade-offs we are willing to make in swapping students between classes. Suppose

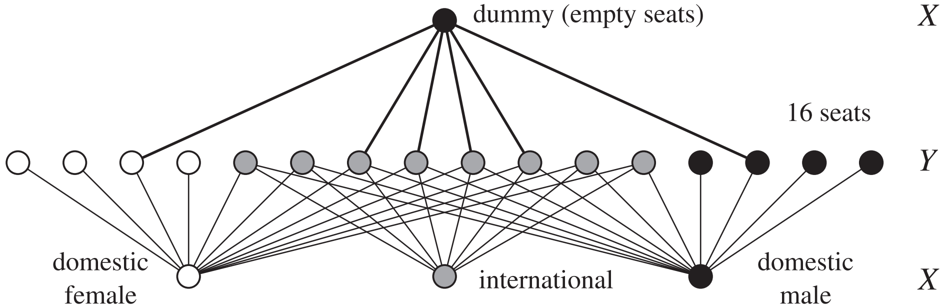


Figure 3 The edges connecting the 16 seats in one class to the four types of student: domestic female, international, domestic male, and dummy students. The edge structure assures that a perfect matching will respect the desired constraints.

that we can move one student up from third choice to second; this reduces the total weight by $\alpha^2 - \alpha$. Moving a student down from first to second adds $\alpha - 1$ to the total. Since $\alpha^2 - \alpha = \alpha(\alpha - 1)$, we see that α represents the number of downward-moving students that balances an upward move of one student; for example, if $\alpha = 3.1$, then moving one student up is an improvement if it causes three or fewer students to move down. Other swap profiles have similar interpretations. Finally, we assign a weight of a^4 to every dummy edge. This large weight ensures that the algorithm always prefers a real student over a dummy student.

This model was a useful tool for attaining a desirable outcome: When there is an assignment that meets all constraints, the algorithm is guaranteed to find it, and it is very fast. But it can and does happen that the graph does not have a perfect matching (a matching that covers all vertices). This will happen if the constraints cannot all be met: Perhaps there is a course that only 3 men have chosen as possibilities; or 3 courses that have male preferences with overlap so that not all of them can have 4 men. In such cases, there is no perfect matching in the graph and the algorithm fails. Another difficulty (which arose in 2013) is when student choices are such that the class-size constraint (either the minimum or the maximum) cannot be met. For example, the data might force there to be, say, 12 classes that must increase enrollment to 17. Some of these problems can be handled by setting up a new graph; but that is tedious, and worse, there is no global solution. For example, we cannot set up a graph that will provide at most 12 classes of size 17. Instead, we must specify which particular classes will rise to 17.

Yet another problem arises when a college has a different set of constraints. For example, Beloit College uses our methods, but wishes to avoid having students from the same high school placed in the same class, something that cannot easily be set up in the graph environment. We imagine that some colleges might want to restrict on other demographics, such as state of residence or athletic participation. Such issues (infeasible or complicated constraints) will lead to trouble for the Hungarian Sorting Hat.

This led us to the Integer Linear Programming method, which is easy to set up and can handle a wide variety of constraints. And because ILP, when it halts, finds the true minimum, it will always perform at least as well as the graph theory method.

A Flexible Sorting Hat

The general Linear Programming (LP) problem is to minimize a linear objective function $\sum c_i x_i$ (where c_i are given real numbers) subject to the variables x_i satisfying a collection of linear constraints (inequalities or equalities). There are very efficient algorithms, including the venerable simplex method, that can solve even very large LP problems. The simplex algorithm is similar to Gaussian elimination: a straightforward algebraic process. But for discrete work, we need solutions in integers, and that variation is called Integer Linear Programming (ILP).

The ILP problem has been well studied and is much more difficult than LP. Still, there are various packages for ILP that splice together a variety of methods; they can work surprisingly well on problems with a few thousand variables. A major step in most ILP algorithms is to make repeated (100s or 1000s) calls to LP, using a branch-and-bound approach. A very brief introduction to branch-and-bound is in [**8**]; see [**7**] for more detail. *Mathematica* has an ILP solver available.

So in our case, we need to set up a linear system of inequalities and an objective function so that (a) a solution to the system gives us a workable assignment of students, and (b) minimizing the objective gives an assignment that maximizes student happiness.

The basic variables are easy to set up. Because we will use ILP, all variables are restricted to be integers. Let the students be S_i, $1 \le i \le n$, and the courses C_j, $1 \le j \le m$; let M, F, I denote the indices corresponding to male, female, and international students, respectively. Let the preferences of S_i be P_i, an ordered list of four integers, the indices for courses the student is willing to take. For each course C_j, let X_j be the set of student indices i so that $j \in P_i$; that is, X_j is the set of students having C_j as one of their choices. As noted, student preferences can be problematic; an extreme case would be some $X_j = \varnothing$, but if some X_j is very small, that will also require some manual intervention, or possibly the canceling of a class. We will describe the setup for the constraints as given above. A reader facing different constraints should be able to modify the setup without difficulty.

The variables. For each pair (S_i, C_j) where $j \in P_i$, introduce a variable $x_{i,j}$, which is to be 0 or 1, with a 1 indicating that S_i is assigned to C_j. This contributes $4n$ variables.

The basic constraints. For each variable, $x_{i,j} \in \{0, 1\}$, which arises from the linear constraint $0 \le x_{i,j} \le 1$. Also, each student gets exactly one course: For each $i \le n$, $\sum_{j \in P_i} x_{i,j} = 1$.

The objective function. The objective is to minimize

$$\sum_{i=1}^{n} x_{i,j_{i,1}} + \alpha x_{i,j_{i,2}} + \alpha^2 x_{i,j_{i,3}} + \alpha^3 x_{i,j_{i,4}},$$

where $j_{i,1}, \ldots, j_{i,4}$ are student S_i's four choices and $\alpha > 1$ is the fixed parameter described above.

Course size constraint. There are an upper bound U and a lower bound L on course size: For each $j \le m$, $L \le \sum_{i \in X_j} x_{i,j} \le U$.

Gender constraint. A typical gender constraint might be that each course contains at least four students of each gender: For each $j \le m$, $\sum_{i \in M \cap X_j} x_{i,j} \ge 4$ and $\sum_{i \in F \cap X_j} x_{i,j} \ge 4$.

International constraint. Each course contains at most B international students: For each $j \le m$, $\sum_{i \in I \cap X_j} x_{i,j} \le B$.

The preceding are the ideal constraints, but it can happen that not all of them can be met. One way to handle typical problems is to introduce extra variables. For ease of exposition, we will use $(L, U) = (12, 16)$, but other values can be handled in the same way. Since allowing a course to grow to 17 is common, as is allowing a course to have fewer than 12, let $q_{j,s}$ be a 0-1 variable that indicates that the size of C_j is s, where $9 \le s \le 17$. Assume that four parameters Q_{17}, Q_{11},Q_{10}, Q_9 are given; they represent the maximum number of classes whose size is allowed to be 17, 11, 10, 9, respectively. Now the number of variables is $4n + 4m$. Then the ideal size constraint is augmented by the following:

For each j and s, add the constraint $0 \le q_{j,s} \le 1$; and for each j,

$\sum_j q_{j,17} \le Q_{17}$ (number of courses with 17 students is at most Q_{17});

$\sum_j q_{j,11} \le Q_{11}$ (number of courses with 11 students is at most Q_{11});

$\sum_j q_{j,10} \le Q_{10}$ (number of courses with 10 students is at most Q_{10});

$\sum_j q_{j,9} \le Q_9$ (number of courses with 9 students is at most Q_9);

and

$\sum_{s=9}^{17} q_{j,s} = 1$ (size of C_j is a unique value between 9 and 17);

$\sum_{i \in X_j} x_{i,j} = \sum_{s=9}^{17} s q_{j,s}$ (size of C_j is given by the q-indicator).

A similar enhancement relaxes a gender requirement. Replace the ideal male constraint by the addition of indicator variables $y_{m,j}$ so that a value of 1 means that C_j has m males. The new constraints are $\sum_{m=0}^{17} y_{m,j} = 1$ and $\sum_{m=0}^{17} m y_{m,j} = \sum_{i \in M \cap X_j} x_{i,j}$ (both for all j), and, say, $\sum_j y_{0,j} = \sum_j y_{1,j} = \sum_j y_{2,j} = 0$ and $\sum_j y_{3,j} \le Y_3$, where Y_3 is a new parameter that bounds the number of courses that can be male-deficient by 1. And we can also introduce Y_2 if needed for courses with two males. The variable count is now $4n + 22m$.

Mathematica's ILP implementation can handle a typical setup (there are about 3000 variables in our case) in under five seconds, returning either the optimal solution or a message indicating that the constraints are not feasible. This is so fast that we can run the routine repeatedly to learn what is feasible and what is not. It is when the basic setup is infeasible that ILP is most useful, since the auxiliary bounds (Q_i, Y_3) can be tweaked until a solution is found.

Before using ILP to assign actual students, we tried it on past data sets. The 2004 case was interesting: The profile—1st through 4th percentages of the manual assignment—was (59%, 29%, 11%, 1.4%). The data were such that there had to be a course with only 2 men; other than that, the male constraint could be satisfied. Because that course was easily identified, the graph matching approach would work since we can just adjust the edge set for that one course.

Using ILP, we can vary the settings for the number of male-deficient courses (the Y_3 and Y_2 mentioned earlier), which is how we learn the minimum violations. We can also add constraints of the form "number of 4th choices is at most P_4" as a way of discovering the minimum number of 4th choices (it is 6), and similarly for 3rd choices (it is 8). Using $\alpha = 1.5$ gives the profile (69%, 27%, 3.3%, 1.2%), while using $\alpha = 5.5$ gives (65%, 33%, 1.5%, 1.2%). The latter has fewer in choice 3 and 4, but the former has quite a few more in the first choice. This sort of trade-off is what we typically present to AP. Both are significantly better than the manual result. As a bonus, we learn that the $\alpha = 5.5$ profile is completely optimal as far as minimizing third and fourth choices.

Here is how things worked in July 2013 at Macalester. There were 550 students and 34 courses. But the data set was more difficult than average and there could be no assignment that respected either the upper bound of 16 or the minimum-male requirement of 4. It was easy to learn that there had to be at least twelve courses of size 17 and at least two courses with only 3 males. Because the choice of twelve courses to expand is not clear, there is no easy adjustment to the graph method that can handle this. But ILP has no problem, setting the oversize parameter Q_{17} to 12.

FIGURE 4 shows a chart of the sort we presented to AP in 2013. It shows that twelve courses have size increased to 17 students, no course has fewer than 14 students, and two courses have only 3 males. The preference profile is: 55% into choice 1, 82% into choice 1 or 2, and 3.8% into choice 4. This is definitely worse than other years, but this data set was less than ideal.

The situation in July 2014 was much better. There were 544 students, 35 courses, and all constraints could be satisfied. Our assignment placed 72.4% into their first choice; 26.3% into their second choice; 1.3% into the third choice; and none into a fourth choice. The number of students placed into their top two choices was a record high (98.7%), and this was the first time that Macalester avoided any fourth choices. The performance of our sorting hat in the past two years clearly demonstrates the value of our implementation, in good times and in bad.

In 2013, our ILP method was also used at Beloit College, with 96% getting choice 1 or 2, and 2% getting choice 4. One word of warning though: ILP is an NP-complete problem, and so if the class size grows into the thousands it will likely be too slow.

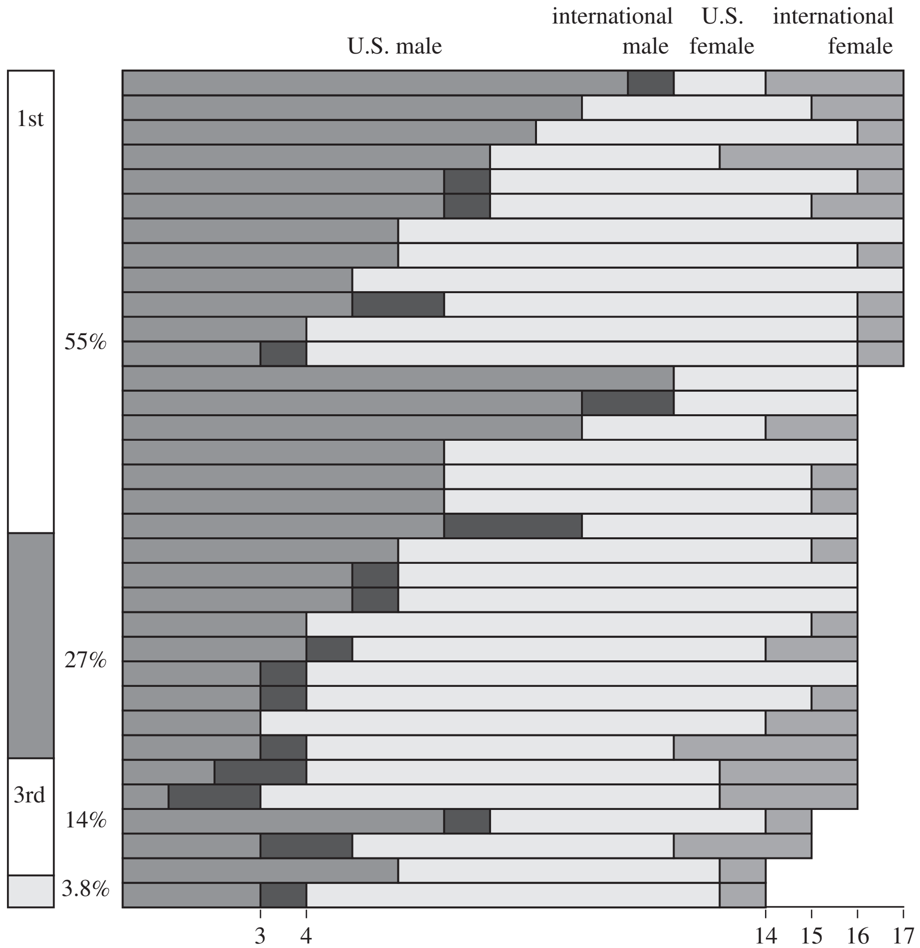


Figure 4 The profile for an assignment produced by ILP for actual 2013 data. Two courses have three males, and twelve courses have 17 students.

Conclusion

This topic is useful to colleges in several ways. It helps the mathematics department, since in an optimization or applied mathematics course the students find the whole discussion fascinating, as it involves something they are very familiar with. They quickly appreciate the potential of both graph theory and ILP to be useful in diverse situations (an especially noteworthy application of both ILP and matching algorithms is in the optimal arrangement of kidney transplants [**1**]). And of course, the methods are directly beneficial to the college administration. It is clear that a sophisticated algorithm will outperform any manual approach to the FYC assignment problem.

Any reader interested in learning how our approach could work at their institution should contact us.

Acknowledgment We are grateful to Paul Campbell for valuable comments on an early draft.

REFERENCES

1. D. J. Abraham, A. Blum, and T. Sandholm, Clearing algorithms for barter exchange markets: Enabling nationwide kidney exchanges, *Proc. 8th ACM Conf. on Electronic Commerce* (2007) 295–303. Available at http://www.cs.cmu.edu/afs/cs/Web/People/sandholm/kidneyExchange.EC07.withGrantInfo.pdf
2. D. Berrett, The many faces of the freshman seminar, *Chronicle of Higher Education*, July 29, 2013. Available at http://chronicle.com/article/The-Many-Faces-of-the-Freshman/140543
3. A. Beveridge and S. Cooke, The mathematical Sorting Hat, *The UMAP Journal of Undergraduate Mathematics and Its Applications* **33:2** (2012) 99–117. Available at https://www.macalester.edu/~abeverid/papers/sortinghat.pdf
4. J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, Elsevier, Amsterdam, 1976.
5. W. Cook, W. Cunningham, W. Pulleyblank, and A. Schrijver, *Combinatorial Optimization*, Wiley, New York, 1998.
6. J. K. Rowling, *Harry Potter and the Sorcerer's Stone*, Scholastic, New York, 1998.
7. P. R. Thie, *An introduction to Linear Programming and Game Theory,* 2nd ed., Wiley, New York, 1988.
8. S. Wagon, An algebraic approach to geometrical optimization, *Math Horizons* (Feb. 2012) 22–27.

Summary We describe the solution to a combinatorial optimization problem that arises in higher education: the assignment of first-year students to introductory seminars. We trace the evolution of our implementation at Macalester College, a small liberal arts school. We first describe how the classic Assignment Problem for bipartite graphs can be used. Then we show how Integer Linear Programming leads to more flexibility, as it allows the optimization to be fine-tuned to handle more challenging data sets.

ANDREW BEVERIDGE is an associate professor of mathematics at Macalester College. He received his doctorate from Yale University under the supervision of László Lovász. Prior to his current position, he held a visiting position at Carnegie Mellon University, and he worked for six years in the software industry.

STAN WAGON is recently retired from Macalester College. His books include *The Banach-Tarski Paradox*, *Mathematica in Action*, and *VisualDSolve*. Other interests include geometric snow sculpture, mountaineering, and mushroom hunting. He is one of the the founding editors of *Ultrarunning* magazine, but now finds that covering long distances is much easier on skis than in running shoes.

Mind Switches in *Futurama* and *Stargate*

RON EVANS
University of California, San Diego
La Jolla, CA 92093-0112
revans@ucsd.edu

LIHUA HUANG
Rutgers University
Piscataway, NJ 08854-8019
lhuang@math.rutgers.edu

"The Prisoner of Benda" [**7, 9**], a 2010 episode of *Futurama*, features a two-body mind-switching machine. Any pair can enter the machine to swap minds, but the machine has the limitation that it will not work more than once on the same pair of bodies. A two-body mind-switching machine with exactly the same limitation is featured in "Holiday" [**4**], a 1999 episode of *Stargate SG-1*. In both episodes, the participants in the mind swapping want to return to their original bodies, but they don't know how. Brilliant characters (Sweet Clyde Dixon in *Futurama*, Captain Samantha Carter in *Stargate SG-1*) save the day by figuring out how to reverse the switching.

We first present a problem arising from *Futurama*. In "The Prisoner of Benda," Amy and the Professor enter the machine and swap minds. Unhappy with their new bodies, they re-enter the machine and attempt to undo the switch. When nothing changes, the Professor recalls that due to cerebral immune response, "Once two bodies have switched minds, they can never switch back." He proposes the idea of employing a third body for temporary mind storage, but he soon realizes that three bodies will not suffice to restore all minds to their original bodies. Agitated, Amy exclaims, "Oh no! Is it possible to get everyone back to normal using four or more bodies?" The Professor responds, "I'm not sure. I'm afraid we need to use . . . math!"

Throughout the episode, nine characters take part in a mind-swapping spree: Fry, Zoidberg, Amy, Hermes, Leela, Professor, Bender, Emperor, and Washbucket. For brevity, call their bodies 1, 2, 3, 4, 5, 6, 7, 8, and 9, respectively. These characters engage in a sequence of seven swaps: First the pair 3, 6 swap minds, then 3, 7, then 5, 6, then 3, 9, then 1, 2, then 8, 9, and finally 4, 5.

After many adventures in each other's bodies, the characters decide that they were better off before the switching. The Professor (in Bender's body) tries in vain to find a way to restore all minds to their original bodies. Sweet Clyde Dixon comes to the rescue with his proof of an inversion theorem displayed on a greenboard. The theorem shows how to undo the switching with a sequence of thirteen swaps, as given in eq. (6) below. Sweet Clyde becomes a hero and the Emperor makes him a duke. Curiously though, the characters actually restore normalcy with a sequence of thirteen swaps *different* from that shown on the greenboard: First s and 1 swap minds, then the pair t, 2, then s, 2, then t, 1, then s, 6, then t, 9, then s, 5, then t, 8, then s, 4, then t, 7, then s, 3, then t, 6, and finally s, 9, where s and t denote the bodies of Globetrotters Sweet Clyde and Bubblegum Tate.

In a popular 2010 video [**5**], Cambridge University mathematician James Grime points out that the switching could be reversed without employing the bodies s and t, using the following nine swaps: First 2 and 3 swap minds, then the pair 1, 9, then 1, 8, then 1, 7, then 1, 6, then 1, 5, then 1, 4, then 1, 3, and finally 2, 9. Grime proceeds to ask if 9 is the *smallest* number of swaps that could accomplish the reversal. In the

Math. Mag. **87** (2014) 252–262. doi:10.4169/math.mag.87.4.252. © Mathematical Association of America

next section, this question is reformulated using mathematical terminology. Then, in Theorem 2, we give an affirmative answer to a general version of the question involving an arbitrary number of bodies.

We next turn to a problem arising from *Stargate SG-1*. In "Holiday," a crisis is created when the aging Machello tricks young handsome Daniel into swapping minds with him. As Jack and Teal'c attempt to help their colleague Daniel by retrieving the mind-switching machine, they blunder into swapping themselves. Unaware that Machello has designed the machine to block a direct reversal, physicist Samantha Carter attempts in vain to rectify the blunder by instructing Jack and Teal'c to grasp the machine's handles in different ways. Meanwhile, Machello is enjoying his holiday in Daniel's body.

Machello later feels guilty about what he has done to Daniel, and tells Samantha, "If I could trade places with him again, I would. But unfortunately, I am the only one who cannot." His last sentence sparks Samantha's idea on how to reverse the switching. She tells Jack, "We're going to have to play a little musical chairs with your bodies." At Samantha's instruction, first the bodies of Teal'c and Daniel swap minds, then Jack and Machello, then Teal'c and Machello, and finally Jack and Daniel. This sequence of four swaps brings everyone back to normal, and Samantha earns the undying gratitude of the Stargate crew. Is 4 the *smallest* number of swaps that can accomplish the reversal? In Theorem 3, we give an affirmative answer to a general version of this question involving an arbitrary even number of bodies.

Mind switching and permutations

In this section, we generalize the aforementioned problems and frame them in the language of group theory. Let S_n denote the group of $n!$ permutations of $\{1, 2, \ldots, n\}$.

We begin with the problem from *Futurama*. Recall that the nine characters engage in a sequence of seven swaps, beginning with the pair 3, 6. We represent this sequence by the right-to-left formal product

$$B_9 := (45) \cdot (89) \cdot (12) \cdot (39) \cdot (56) \cdot (37) \cdot (36), \tag{1}$$

where the factors in eq. (1) are transpositions (2-cycles) in the symmetric group S_9.

In the sequel, the symbol B (with or without a subscript) will always denote a formal product of distinct transposition factors in S_n, representing a sequence of swaps that one aims to reverse. For brevity, "transposition factors" will simply be called "factors." When the factors of B are written in reverse order, the resulting formal product will be denoted by B^{-1}. The product B is an element in the group of words on the alphabet of transpositions.

One can view B as a permutation in S_n via right-to-left multiplication in S_n of its factors. This permutation will be denoted by $\sigma(B)$. Every permutation in S_n can be expressed as a product of nontrivial disjoint cycles [**1**, Theorem 2.3.5], and we will always express permutations such as $\sigma(B)$ in this form. For example, the product $B = (12) \cdot (23)$ effects the 3-cycle $\sigma(B) = (123) \in S_3$. This 3-cycle represents the permutation that sends 1's mind to 2, 2's mind to 3, and 3's mind to 1. From eq. (1),

$$\sigma(B_9) = (12)(3456789). \tag{2}$$

The mind-switching history is encoded in the product B, but not generally in the permutation $\sigma(B)$. To illustrate this, consider the two products $B = (12) \cdot (23)$ and $B' = (23) \cdot (13)$. The product B tells us that the pair $1, 2$ swapped right after the pair $2, 3$, while B' tells us that $2, 3$ swapped right after $1, 3$. Although B and B'

are completely different formal products, they effect the same permutation $\sigma(B) = \sigma(B') = (123)$. This permutation conveys no information that distinguishes between the distinct switching histories encoded by B and B'.

The participants in *Futurama*'s mind-swapping spree wish to undo B_9; that is, they want to reverse the switching in order to bring all minds back to their original bodies. To undo a general product of transpositions B representing a sequence of mind swaps, one needs to construct a formal product Q of distinct transpositions such that

(a) the products Q and B have no factor in common; and

(b) $\sigma(Q) = \sigma(B)$.

Then $\sigma(Q^{-1} \cdot B)$ equals the identity permutation and Q^{-1} undoes B. To reverse the switching as efficiently as possible, we want to undo B using a product Q that has the smallest possible number of factors. Any such Q is said to be "optimal" or "best possible." In the sequel, the symbol Q (with or without a subscript) will always denote a formal product of distinct transpositions.

For the product B_9 in eq. (1), which represents the sequence of mind swaps in "The Prisoner of Benda," Grime [**5**] observed that $\sigma(B_9) = \sigma(Q_9)$, where

$$Q_9 := (23) \cdot \big((19) \cdot (18) \cdot (17) \cdot (16) \cdot (15) \cdot (14) \cdot (13)\big) \cdot (29). \tag{3}$$

The product Q_9 has no factor in common with the product B_9. Thus Q_9^{-1} serves to undo B_9 using nine transpositions. Grime asked if Q_9 is best possible. We will give an affirmative answer to a more general question.

For $n \geq 4$, let B_n be any product of distinct transpositions representing a sequence of mind swaps with the following two properties:

(a) Bodies 1 and 2 swap with each other but not with any other body, as in eq. (1); and

(b) As in eq. (2),

$$\sigma(B_n) = (12)(345 \cdots n). \tag{4}$$

Then, analogous to the case $n = 9$ discussed in the preceding paragraph, we have $\sigma(B_n) = \sigma(Q_n)$, where

$$Q_n := (23) \cdot \big((1n) \cdots (15) \cdot (14) \cdot (13)\big) \cdot (2n). \tag{5}$$

The product Q_n has no factor in common with the product B_n. Thus Q_n^{-1} serves to undo B_n using n transpositions. Is Q_n best possible? For $n = 9$, this is Grime's question. Theorem 2 gives an affirmative answer for each $n \geq 4$.

"The Prisoner of Benda" was written by Ken Keeler [**10**, p. 207], who earned a Ph.D. in applied mathematics at Harvard University in 1990. For the show, Keeler presented an algorithm [**7**] that is designed to reverse any sequence of swaps, that is, it is designed to undo any product B. We gave a best possible algorithm optimizing Keeler's in [**3**, Theorem 1]. Both algorithms depend only on the permutation $\sigma(B)$, not on the product B that effected $\sigma(B)$; in other words, both are independent of the switching history that effected the ultimate permutation. These algorithms require the entries in $\sigma(B)$ to switch only with bodies in the set $\{x, y\}$, where x and y are "outsiders" who did not participate in the initial mind-swapping spree. Because these algorithms are designed for situations where the switching history is forgotten, it should come as no surprise that they may not be optimal in situations where the switching history is known. For example, Keeler's algorithm reverses B_9 with the product of thirteen transpositions given by

$$(x1) \cdot (y2) \cdot (x2) \cdot (y1) \cdot (x3) \cdot (x4) \cdot (x5) \cdot (x6) \cdot (x7) \cdot (x8) \cdot (y9) \cdot (x9) \cdot (y3), \tag{6}$$

while by eq. (3), B_9 can be undone with only nine transpositions. We consider further comparisons of this kind in the final section.

We next look at the problem from *Stargate SG-1*. For brevity, denote the bodies of Teal'c, Jack, Machello, and Daniel by 1, 2, 3, and 4, respectively. Recall that Daniel initially swapped minds with Machello, after which Teal'c swapped minds with Jack. This sequence of two mind swaps can be represented by the product

$$H_4 := (12) \cdot (34). \tag{7}$$

Samantha Carter observed that $\sigma(H_4) = \sigma(Q_4)$, where

$$Q_4 := (24) \cdot (13) \cdot (23) \cdot (14). \tag{8}$$

The products H_4 and Q_4 have no factor in common. Thus Q_4^{-1} serves to undo H_4 using four transpositions. It follows easily from parity considerations (as in Lemma 2 below) that H_4 cannot be undone with fewer than four transpositions, so Q_4 is best possible.

The product in eq. (7) is a special case of the product

$$H_{2r} := (12) \cdot (34) \cdots (2r-1, 2r), \quad r \geq 1. \tag{9}$$

Observe that

$$\sigma(H_{2r}) := (12)(34) \cdots (2r-1, 2r),$$

since the factors of H_{2r} are already disjoint. We proceed to extend the problem of undoing H_4 to that of undoing H_{2r}. The switching history now is a sequence of r disjoint mind swaps represented by eq. (9). We are interested in finding an optimal product Q for which Q^{-1} undoes H_{2r}. It turns out that such a product Q needs no outsiders, except in the case $r = 1$.

First consider the case $r = 1$. Using the outsiders 3 and 4, we have

$$\sigma(Q_2) = \sigma(H_2) = (12),$$

where

$$Q_2 := (34) \cdot (23) \cdot (14) \cdot (13) \cdot (24). \tag{10}$$

Since the transposition (12) is not a factor of the product Q_2, it follows that Q_2^{-1} serves to undo H_2 using five transpositions. We can show directly that H_2 cannot be undone with fewer than five transpositions. For suppose that (12) were equal to a product of $u < 5$ distinct transpositions, none equal to (12). Then since (12) is an odd permutation, we would have $u = 3$ (again by Lemma 2), so that some product of four distinct transpositions would equal the identity. This is easily seen to be impossible.

Having already discussed the case $r = 2$, we next consider the case $r = 3$. Observe that

$$\sigma(Q_6) = \sigma(H_6) = (12)(34)(56),$$

where

$$Q_6 := (15) \cdot (25) \cdot (35) \cdot (46) \cdot (45) \cdot (16) \cdot (13). \tag{11}$$

The product Q_6 has no factor in common with $H_6 = (12) \cdot (34) \cdot (56)$. Thus Q_6^{-1} serves to undo H_6 using seven transpositions.

For general $r > 3$, we can undo H_{2r} as follows. If r is even, undo the first two factors of H_{2r} using four transpositions as in eq. (8), then repeat the process for the next two factors of H_{2r}, and so forth. In this way, we undo H_{2r} with $2r$ transpositions. If r is odd, undo the first three factors of H_{2r} using seven transpositions as in eq. (11), then undo consecutive pairs of remaining factors as in the case where r is even. In this way, we undo H_{2r} with $2r+1$ transpositions. In summary, for any $r \geq 2$, the product H_{2r} can be undone with $2r + \epsilon_r$ transpositions, where

$$\epsilon_r = \begin{cases} 0, & \text{if } 2 \mid r \\ 1, & \text{if } 2 \nmid r. \end{cases} \tag{12}$$

Is this best possible? Theorem 3 gives an affirmative answer for each $r \geq 2$. In the exceptional case $r = 1$, we've already seen that H_2 can be undone with five transpositions, and this is best possible.

The proofs of Theorems 2 and 3 depend on Theorem 1, which may be of independent interest because it answers the following question about a permutation σ expressed as a product of nontrivial disjoint cycles. "When writing σ as a product of distinct transpositions, none occurring as factors in the disjoint cycle representation of σ, what is the smallest number of transpositions that can be used?"

Optimality theorems

Let σ be a permutation expressed as a product of $m \geq 1$ nontrivial disjoint cycles, and let $n \geq 2$ denote the number of entries in σ. A paper by Mackiw [**8**] opens with this question: When writing σ as a product of transpositions, what is the smallest number of transpositions that can be used? The well-known answer is $n - m$; see [**8**], [**6**, Theorem 4].

The following theorem answers a refinement of this question when $n > 2$. For brevity, if a transposition τ appears as a factor in the disjoint cycle factorization of σ, we will simply say "τ occurs in σ."

THEOREM 1. *Let σ be a permutation expressed as a product of $m \geq 1$ nontrivial disjoint cycles, and let $n > 2$ denote the number of entries in σ. Suppose that $\sigma = \sigma(Q)$, where Q is a product of distinct transpositions, none occurring in σ. Then the smallest number M of transposition factors that Q can have is*

$$M = n - m + R + \epsilon_R, \tag{13}$$

where R is the number of transpositions occurring in σ, and ϵ_R is as defined in eq. (12).

It is interesting to note that M depends only on the number of transposition factors, the number of cycles, and the number of entries in σ, but not on any additional information about the cycle structure of σ. For example, either $\sigma = (12)(345)(678)(9abcde)$ or $\sigma = (1234)(56)(789a)(bcde)$ would yield the same value of M.

Theorem 1 does not hold for $n = 2$, since then $M = 5$ by the argument following eq. (10). It can be shown that Theorem 1 would be valid for all $n \geq 2$ if we dropped the requirement that the factors of Q be distinct. In particular, for the case $n = 2$, note that $(12) = (23)(13)(23)$.

We prove Theorem 1 in the next section, using the following two well-known results from graph theory and group theory. Proofs of these results may be found in [**2**, Theorem 11.2.1, p. 163] and [**1**, pp. 82, 149], respectively.

LEMMA 1. *A connected graph on N vertices has at least $N - 1$ edges.*

LEMMA 2. *If the product of k transpositions in the symmetric group S_n equals the identity, then k must be even.*

Theorems 2 and 3 below solve the optimality problems arising from *Futurama* and *Stargate SG-1* that were posed at the start of the paper. To prove Theorem 2, we will apply Theorem 1 with σ chosen to be $\sigma(B_n)$. In this case, the lone transposition (12) occurring in σ is also a factor of B_n. To prove Theorem 3, we will apply Theorem 1 with σ chosen to be $\sigma(H_{2r})$. In this case, all transpositions occurring in σ are factors of H_{2r}. We are thereby able to exploit the switching histories that effected $\sigma(B_n)$ and $\sigma(H_{2r})$, even though no switching history is involved in Theorem 1 per se.

THEOREM 2. *Let $n \geq 4$. Then the product B_n defined above eq.* (5) *cannot be undone with fewer than n transpositions.*

Proof. Recall from eq. (4) that $\sigma(B_n) = (12)(345\cdots n)$. Suppose that $\sigma(B_n) = \sigma(Q)$ for some product Q of distinct transpositions, none equal to a factor of B_n. We need to show that Q has at least n factors. In the case $n = 4$, the result follows by the argument below eq. (7) with B_4 in place of H_4. Now suppose that $n \geq 5$. By definition, B_n has the factor (12); thus the product Q cannot have (12) as a factor. It follows that Q is a product of distinct transpositions none occurring in $\sigma(B_n)$. We can therefore apply Theorem 1 with $\sigma = \sigma(B_n)$ to conclude that the number of factors in Q is at least

$$M = n - m + R + \epsilon_R = n - 2 + 1 + 1 = n. \qquad \blacksquare$$

THEOREM 3. *Let $r \geq 2$. Then the product H_{2r} defined in eq.* (9) *cannot be undone with fewer than $2r + \epsilon_r$ transpositions.*

Proof. Suppose that $\sigma(H_{2r}) = \sigma(Q)$ for some product Q of distinct transpositions, none equal to a factor of H_{2r}. Therefore Q is a product of distinct transpositions none occurring in $\sigma(H_{2r})$. Applying Theorem 1 with $\sigma = \sigma(H_{2r})$ and $R = r$, we conclude that the number of factors in Q is at least

$$M = n - m + r + \epsilon_r = 2r - r + r + \epsilon_r = 2r + \epsilon_r. \qquad \blacksquare$$

Proof of Theorem 1

With $R = r$, we may express the permutation σ in Theorem 1 as the following product of disjoint cycles:

$$\sigma = \sigma(H_{2r})C_1 \cdots C_{m-r}, \tag{14}$$

where H_{2r} is defined in eq. (9) and the C_i are disjoint cycles of length $\ell_i > 2$. (If $m = r$, then σ is interpreted as $\sigma(H_{2r})$.) Our object is to prove eq. (13). We may suppose that $r \geq 1$, since if no transposition occurs in σ, then $M = n - m$ as in [**8**].

We proceed to show that σ has parity $n - m + r + \epsilon_r$, by which we mean that the permutation σ is even or odd according as the integer $n - m + r + \epsilon_r$ is even or odd. Each C_i equals a product of $\ell_i - 1$ transpositions; for example, $(abcdef) = (ab)(bc)(cd)(de)(ef)$. Therefore $C_1 \cdots C_{m-r}$ has parity

$$\sum_{i=1}^{m-r}(\ell_i - 1) \;=\; (n - 2r) - (m - r) = n - m - r.$$

Since $\sigma(H_{2r})$ is a product of r transpositions, it follows from eq. (14) that σ has parity $n - m$. Equivalently, σ has parity $n - m + r + \epsilon_r$, since $r + \epsilon_r$ is even.

Our object is to prove that $M = n - m + r + \epsilon_r$ for $n > 2$ and $r \geq 1$. We begin by showing that

$$M \leq n - m + r + \epsilon_r. \tag{15}$$

To accomplish this, it must be shown that $\sigma = \sigma(Q)$ for some product Q of $n - m + r + \epsilon_r$ distinct transpositions, none occurring in σ.

First consider the case $r = 1$. By eqs. (4) and (5), the permutation $(12)C_1$ equals a product of $2 + \ell_1$ distinct transpositions, none equal to the transposition (12). Thus $\sigma = (12)C_1 \cdots C_{m-1}$ equals a product of

$$2 + \ell_1 + \sum_{i=2}^{m-1}(\ell_i - 1) = n - m + 2$$

distinct transpositions, none equal to the transposition (12). This proves eq. (15) in the case $r = 1$.

Next let $r \geq 2$. We have seen that for $r \geq 2$, $\sigma(H_{2r})$ equals a product of $2r + \epsilon_r$ distinct transpositions, none equal to a factor of H_{2r}. Thus $\sigma = \sigma(Q)$ for a product Q of

$$2r + \epsilon_r + \sum_{i=1}^{m-r}(\ell_i - 1) = n - m + r + \epsilon_r$$

distinct transpositions, none occurring in σ. This completes the proof of eq. (15). Note that Q has been constructed such that every entry in Q is an entry in σ. This is in contrast to the case $n = 2$, as we have seen.

It remains to prove the reverse inequality

$$M \geq n - m + r + \epsilon_r. \tag{16}$$

We need only prove

$$M \geq n - m + r; \tag{17}$$

indeed, eqs. (16) and (17) are equivalent by Lemma 2, since as was noted above, σ has parity $n - m + r + \epsilon_r$. Supposing that $\sigma = \sigma(Q)$ for a product Q of w distinct transpositions, none occurring in σ, our goal is now to show that $w \geq n - m + r$.

To illuminate the exposition, we will use the running example

$$\sigma = (12)(34)(56)(78)(9ab)(cde) = \sigma(Q),$$

with Q equal to the product

$$(15)\cdot(89)\cdot(25)\cdot(7b)\cdot(4x)\cdot(35)\cdot(7a)\cdot(46)\cdot(79)\cdot(45)\cdot(8b)\cdot(16)\cdot(13)\cdot(cd)\cdot(de)\cdot(3x),$$

where the symbols a, b, c, d, e, x stand for the entries 10, 11, 12, 13, 14, 15. In this example, $n = 14$, $r = 4$, $m = 6$, and $w = 16$.

Let G be a graph whose vertex set $V(G)$ is the set of entries in Q, and whose edges $[ij]$ correspond to the w factors (ij) of Q. The graph G for the product Q in our example is illustrated in FIGURE 1. Since $\sigma = \sigma(Q)$, the set $V(G)$ contains all the entries in σ, but $V(G)$ may also contain entries in Q outside of σ. In our example, x is the only entry in Q outside of σ.

For each i with $1 \leq i \leq m - r$, G has a connected component G_i such that $V(G_i)$ contains all of the entries in the cycle C_i. Let J denote the union of the components G_i.

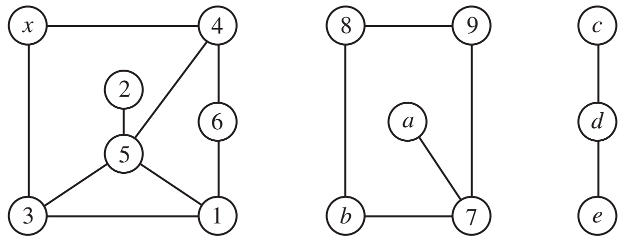


Figure 1 Graph G for the running example

(View J as empty if $m = r$.) In our example, J is the union of two components G_1, G_2 with $V(G_1) = \{7, 8, 9, a, b\}$, $V(G_2) = \{c, d, e\}$, so $V(J) = \{7, 8, 9, a, b, c, d, e\}$.

The set $\{1, 2, \ldots, 2r\} \subset V(G)$ can be written as a disjoint union

$$\{1, 2, \ldots, 2r\} = A \cup Z, \tag{18}$$

where $A \subset V(J)$ and Z is disjoint from $V(J)$. For the cardinalities, write $\alpha := |A|$, so that $|Z| = 2r - \alpha$. In our example, $A = \{7, 8\}$ with $\alpha = 2$ and $Z = \{1, 2, 3, 4, 5, 6\}$.

Since $V(J)$ contains the $n - 2r$ entries in the cycles C_i, we have $|V(J)| \geq n - 2r + \alpha$. By Lemma 1, each G_i contains at least $|G_i| - 1$ edges. Therefore, since the graph J is the disjoint union of at most $m - r$ distinct components G_i, the number of edges in J is at least

$$(n - 2r + \alpha) - (m - r) = n - m + \alpha - r.$$

It follows that Q has at least $n - m + \alpha - r$ distinct transposition factors whose entries are all in $V(J)$. In particular, $w \geq n - m + \alpha - r$, but this does not quite accomplish our goal of showing that $w \geq n - m + r$. Thus we seek additional transposition factors of Q, this time with entries *outside* of $V(J)$.

With the aim of obtaining these additional factors of Q, we will construct a set E with $Z \subset E \subset V(G)$ such that E is disjoint from $V(J)$ and the elements of E fill at least $2|E|$ slots in Q. (Note that any element of E not in Z is necessarily outside of σ.) Since $|E| \geq |Z|$, we will obtain the desired lower bound

$$w \geq (n - m + \alpha - r) + |E| \geq (n - m + \alpha - r) + |Z| = n - m + r.$$

If each of the elements in Z fills at least two slots in Q, then we may take $E = Z$ and the proof is complete. It remains to construct E in the case where Z contains "singletons," that is, elements that occur only once as entries in Q. Note that a singleton corresponds to a vertex of degree 1 in $V(G)$. In our example, 2 is the only singleton in $Z = \{1, 2, 3, 4, 5, 6\}$.

We proceed to specify the elements that will make up the desired set E defined in eq. (19) below. A singleton $b_1 \in Z$ is paired in Q with some $g \in V(G)$, that is, Q has a factor (gb_1). In our example, $(gb_1) = (25)$. Note that g cannot be a singleton, otherwise (gb_1) would be a factor of Q that occurs in σ. Since $b_1 \notin V(J)$, we have $g \notin V(J)$. Let $S(g) := \{b_1, \ldots, b_k\}$ be the set of all singletons in Z that are paired with g, and let

$$(gb_1),\ (gb_2),\ \ldots,\ (gb_k)$$

be transpositions appearing in that same left-to-right order among the factors of Q. In our example, $k = 1$.

Since no factor of Q occurs in σ and $b_1 \in Z$, the permutation σ cannot map the singleton b_1 to g. Thus g occurs as an entry in Q to the left of the factor (gb_1). Similarly, σ cannot map g to the singleton b_k, so g occurs as an entry in Q to the right of (gb_k). Therefore g occurs at least $k+2$ times as an entry so that, all together, the $k+1$ elements in

$$U(g) := \{b_1, \ldots, b_k, g\} = S(g) \cup \{g\}$$

fill at least $2(k+1) = 2|U(g)|$ slots in Q. In our example, (15) occurs to the left of the factor $(gb_1) = (25)$ in Q, both (35) and (45) occur to the right of (25), and the two elements in $U(5) = \{2, 5\}$ together fill five slots in Q.

If Z contains another singleton $b'_1 \notin S(g)$, then repeat this procedure with an element $g' \in V(G)$ that is paired with b'_1. Clearly $g' \neq g$. Note that $S(g')$ is disjoint from $S(g)$, since an element common to both sets would have to occur at least twice as an entry in Q. Thus $U(g')$ is disjoint from $U(g)$. Repeat the procedure again and again until all the singletons in Z have been exhausted.

Let Z^* denote the set of all non-singletons in Z that are not in the set $\{g, g', \ldots\}$. The elements of Z^* fill at least $2|Z^*|$ slots in Q. Define E to be the disjoint union

$$E = Z^* \cup U(g) \cup U(g') \cup \cdots . \tag{19}$$

In our example, $E = Z$ (but E would strictly contain Z in an example where the set $\{g, g', \ldots\}$ contained an element outside of σ). The set E in eq. (19) satisfies the required conditions, since

$$E = Z \cup \{g, g', \ldots\} \subset V(G),$$

E is disjoint from $V(J)$, and the elements of E fill at least $2|E|$ slots in Q. ■

Conclusion and related problems

Let $\sigma(B)$ be a known permutation in S_n that corresponds to a product B representing a (possibly forgotten) switching history. This section begins with a summary of optimal ways for undoing B when

(a) B is unknown,
(b) $B = H_{2r}$, as defined in eq. (9),
(c) $B = B_n$, defined above eq. (5).

We then discuss the situation for some other classes of products B. Our examples illustrate how dramatically different the optimal solutions can be for products B and B' representing different switching sequences, even when B and B' effect the same permutation $\sigma(B) = \sigma(B')$.

Assume that $\sigma(B)$ has n entries and m nontrivial disjoint cycles. Let $j(B)$ denote the number of factors in an optimal product Q for which Q^{-1} undoes B. If B represents a *forgotten* switching history, then by [**3**, Theorem 1],

$$j(B) = n + m + 2, \tag{20}$$

and Q requires two outsiders. In cases where there is full information about B, we can often reduce the right side of eq. (20). For example, if $B = H_n = H_{2r}$ with $r \geq 2$, then Theorem 3 and the statement containing eq. (12) show that

$$j(B) = n \quad \text{or} \quad j(B) = n+1 \quad \text{according as } r \text{ is even or odd,}$$

and no outsiders are needed. Even if only partial information is available about B, for instance when $B = B_n$, we may be able to reduce the right side of eq. (20). For example, Theorem 2 and eq. (5) show that

$$j(B_n) = n, \quad \text{for } n \geq 4, \tag{21}$$

and no outsiders are needed.

Let B'_n denote the product Q_n defined in eq. (5). Then $\sigma(B'_n) = \sigma(B_n)$. We can reduce the right side of eq. (21) upon replacing B_n by B'_n, as follows:

$$j(B'_n) = n - 2, \quad \text{for } n \geq 4. \tag{22}$$

The idea behind eq. (22) is to undo B'_n with the product

$$(12) \cdot (3n) \cdot (4n) \cdots (n-1, n).$$

We can show that this way of undoing B'_n is best possible by using Lemma 2 and [**3**, Lemma 1(i),(ii)]. For an example with $n = 5$, we have

$$B'_5 = (23) \cdot (15) \cdot (14) \cdot (13) \cdot (25) \quad \text{and} \quad \sigma(B'_5) = \sigma(B_5) = (12)(345),$$

and $j(B'_5) = 3$, since $(12)(35)(45)$ undoes B'_5.

Yet another product B with $\sigma(B) = (12)(345)$ is

$$B = (25) \cdot (45) \cdot (35) \cdot (14) \cdot (24).$$

Since $\sigma(B) = \sigma(Q)$ for $Q := (23) \cdot (12) \cdot (15) \cdot (13) \cdot (34)$, Q^{-1} serves to undo B with five transpositions. We leave it as an exercise for the reader to show that this is best possible, that is, $j(B) = 5$.

We proceed to give the values of $j(B)$ for three different classes of products B that effect the same n-cycle $\sigma(B) = (123 \cdots n)$. Again, these examples illustrate how sensitive the optimal solutions can be to the switching history. It would be interesting to find $j(B)$ for other natural classes of products B, and to determine how many outsiders are required for undoing these B.

(a) If

$$B = (12) \cdot (23) \cdot (34) \cdots (n-1, n) \quad \text{with } n \geq 4,$$

then $\sigma(B) = (123 \cdots n)$ and B can be undone with $n + 1$ transpositions, without outsiders. Moreover, by [**3**, Theorem 2], $n + 1$ is best possible, that is, $j(B) = n + 1$.

(b) If

$$B = (n, n-1) \cdots (n3) \cdot (n2) \cdot (n1) \quad \text{with } n \geq 3,$$

then $\sigma(B) = (123 \cdots n)$ and B can be undone with $n + 1$ transpositions, where one outsider is necessary and sufficient. Moreover, by [**3**, Theorem 3], $n + 1$ is best possible, that is, $j(B) = n + 1$.

(c) In items (a) and (b) above, B has only $n - 1$ factors. If B is a product of all $(n^2 - n)/2$ transpositions in S_n, then two outsiders are necessary and sufficient to undo B, and by [**3**, Theorem 1], $j(B) = n + m + 2$. As shown in [**3**], when n is congruent to 1 modulo 4, it is not hard to give a recipe for constructing such B in S_n with $\sigma(B) = (123 \cdots n)$. For an example in S_5, consider the product of all ten transpositions

$$B = (54) \cdot (53) \cdot (52) \cdot (51) \cdot (12) \cdot (23) \cdot (14) \cdot (13) \cdot (24) \cdot (34);$$

then $\sigma(B) = (12345)$ and $j(B) = 8$.

Examples of two products B and B' that effect the same 4-cycle $\sigma(B) = \sigma(B') = (1234)$ are

$$B = (24) \cdot (23) \cdot (14) \quad \text{and} \quad B' = (23) \cdot (12) \cdot (34) \cdot (14) \cdot (24).$$

It is easy to see that $(34) \cdot (12) \cdot (13)$ undoes B and $j(B) = 3$. We close with another fun problem for the reader: Show that two outsiders are necessary and sufficient to undo B', and $j(B') = 7$.

REFERENCES

1. J. Beachy and W. Blair, *Abstract Algebra*, 3rd ed., Waveland Press, Long Grove, IL, 2006.
2. P. J. Cameron, *Combinatorics: Topics, Techniques, Algorithms*, Cambridge University Press, Cambridge, 1994.
3. R. J. Evans, L. Huang, and T. Nguyen, Keeler's theorem and products of distinct transpositions, *Amer. Math. Monthly* **121** (2014) 136–144. http://dx.doi.org/10.4169/amer.math.monthly.121.02.136
4. Gateworld, synopsis of *Stargate SG-1*'s episode number 217 entitled "Holiday," first aired February 5, 1999 during season 2, available at http://www.gateworld.net/sg1/s2/217.shtml.
5. J. Grime, Futurama and Keeler's theorem: original edit, video exposition of Keeler's theorem as presented in *Futurama*'s 2010 episode "The Prisoner of Benda," uploaded on August 22, 2010, available at http://www.youtube.com/watch?v=8M4dUj7vZJc.
6. D. Higgs and P. de Witte, On products of transpositions and their graphs, *Amer. Math. Monthly* **86** (1979) 376–380. http://dx.doi.org/10.2307/2321098
7. The Infosphere, synopsis of *Futurama*'s episode number 98 entitled "The Prisoner of Benda," aired on August 19, 2010, available at http://theinfosphere.org/The_Prisoner_of_Benda.
8. G. Mackiw, Permutations as products of transpositions, *Amer. Math. Monthly* **102** (1995) 438–440. http://dx.doi.org/10.2307/2975036
9. T. Phillips, Original math on *Futurama* (2010), available at http://www.ams.org/news/math-in-the-media/10-2010-media.
10. S. Singh, *The Simpsons and Their Mathematical Secrets*, Bloomsbury, New York, 2013.

Summary We generalize two mind-switching problems that arise in connection with the popular sci-fi television series *Futurama* and *Stargate SG-1*. Optimal solutions to these problems are found by answering the following question about a permutation σ expressed as a product of nontrivial disjoint cycles. "When writing σ as a product of distinct transpositions, none occurring as factors in the disjoint cycle representation of σ, what is the smallest number of transpositions that can be used?"

RON EVANS received a B.S. from the University of Michigan in 1967 and a Ph.D. from the University of Illinois in 1974. After a two-year instructorship at the University of Wisconsin, he joined the UCSD mathematics department in 1975. His main interest is number theory, with recent work focusing on character sum evaluations with connections to Hecke eigenfunctions. He is coauthor, with his Ph.D. advisor Bruce C. Berndt and Kenneth S. Williams, of *Gauss and Jacobi Sums* (Wiley, 1998).

LIHUA HUANG obtained her B.S. in mathematics from the University of California at San Diego in 2012. Her research interests include group theory and several complex variables. She has presented her research at conferences held by Ohio State, Cal Poly Pomona, Penn State, UCSD, and the 2013 Joint Mathematics Meetings in San Diego. In 2012, she became a graduate student in mathematics at Rutgers University, where she received a GAANN Fellowship.

Gabriel's Horn: A Revolutionary Tale

VINCENT COLL
Lehigh University
Bethlehem, PA 18015
vec208@lehigh.edu

MICHAEL HARRISON
The Pennsylvania State University
University Park, PA 16802
mah5044@gmail.com

Gabriel's Horn is the surface generated when the graph of the function $f(x) = x^{-1}$, defined for $x \geq 1$, is revolved around the x-axis as in FIGURE 1. The Horn was discovered in 1641 by Evangelista Torricelli, who established its celebrated properties, infinite surface area and finite volume—which we will refer to together as the *Horn Property*. The proper name Gabriel refers to the Archangel Gabriel, who in some religious traditions is viewed as the messenger of God, heralding the End of Days with a trumpet blast. Coupling the divine with the infinite completes the metaphor for the Horn, which is elsewhere called *Torricelli's trumpet* or the *infinite paint can*. The latter description gives rise to the "painter's paradox": How can a paint can full of paint contain insufficient liquid to coat its interior?

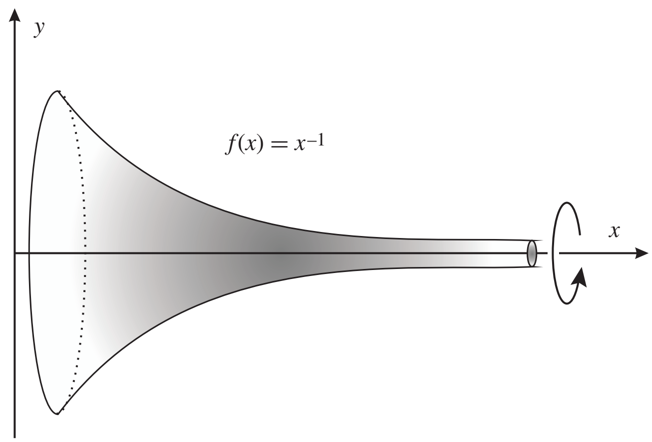


Figure 1 Gabriel's Horn

When first announced, the paradoxical nature of the Horn caused a furor, evoking spirited philosophical discussions on the nature of the infinite and leading the English philosopher Thomas Hobbes to exclaim that believing Torricelli was madness: "...to understand this for sense, it is not required that a man should be a geometrician or a logician, but that he should be mad" [**9**].

The novelty of the Horn spawned other seemingly fantastic examples, such as the drinking vessel of Huygens and deSluze [**10**], and even foreshadowed some of the curious properties of fractal sets such as *Koch's snowflake*, which encloses a finite area with an infinite perimeter [**17**].

Math. Mag. **87** (2014) 263–275. doi:10.4169/math.mag.87.4.263. © Mathematical Association of America

Torricelli's own words express his wonder at the example, clearly illuminating the difficulty that mathematicians of his day had with infinitesimals and quadrature, concepts which had kept no less a mind than Archimedes' from fully discovering the calculus a millennia earlier.

> It may seem incredible that although this solid has an infinite length, nevertheless none of the cylindrical surfaces we considered has an infinite length but all of them are finite.
>
> *Evangelista Torricelli*

Here, "cylindrical surfaces" refers to nested right circular cylinders whose axes are the x-axis and that are inscribed inside the Horn. Each of these cylinders has the same finite surface area and so too, it would seem, must their degenerate limit. However, the degenerate limit is the infinitely long x-axis. Without the language of calculus, Torricelli was unable to resolve this apparent paradox [**12**]. A more thorough explanation of Torricelli's approach can be found in a recent article in this MAGAZINE [**2**].

The paradox of the infinite paint can disappears when we realize that it is the difference in dimensions that creates the illusion of impossibility. We cannot paint the two-dimensional Horn with a uniformly thick coating of three-dimensional paint. For the Horn, we would have to reduce the thickness as we paint to the right. If we are careful, and reduce the thickness quickly enough, any amount of paint will do: We could use an eyedropper full of paint to coat the can.

Given the language of calculus, it is easy to establish the Horn's volume V and surface area A:

$$V = \int_1^\infty \pi [f(x)]^2\, dx = \pi \int_1^\infty \frac{1}{x^2}\, dx = \pi,$$

and

$$\begin{aligned} A &= \int_1^\infty 2\pi f(x)\sqrt{1 + [f'(x)]^2}\, dx \\ &= 2\pi \int_1^\infty \frac{\sqrt{1 + \frac{1}{x^4}}}{x}\, dx > 2\pi \int_1^\infty \frac{1}{x}\, dx = \infty. \end{aligned}$$

Evidently, the critical issue is one of convergence. The phenomenon exhibited by Gabriel's Horn is closely related to the convergence of certain p-series. We note that the defining curve, $f(x) = x^{-1}$, is one among many that will generate an object with the finite volume, infinite surface area property. For example, the profile function $g(x) = x^{-p}$, for any p satisfying $\frac{1}{2} < p \leq 1$, will also work. As an interesting aside, Fleron [**8**] uses step functions to create a discrete version of Gabriel's Horn—*Gabriel's wedding cake*—which we "can eat but cannot frost" (FIGURE 2).

Despite the age and celebrity of Gabriel's Horn, we find no evidence that higher-dimensional analogues were investigated before the last decade. In a 2004 article [**7**] in this MAGAZINE, Eisenberg provides an intuitive introduction to 3-dimensional hypersurfaces of revolution in 4-dimensional Euclidean space, including both heuristic arguments and analytic proofs of the associated volume formulas. He notes that Gabriel's Horn generalizes neatly to this setting by finding 3-dimensional hypersurfaces of revolution with the Horn Property—infinite "surface area" (properly, 3-volume) enclosing a region of finite 4-volume. Three years later, Aberra and Agrawal [**1**] extended these results to arbitrary dimensions $n \geq 2$ by presenting the volume formulas associated with an n-dimensional hypersurface of revolution. There, the evident pattern

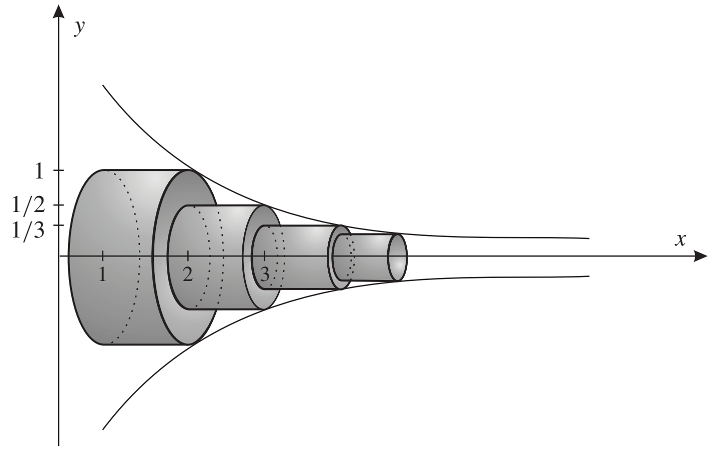


Figure 2 Gabriel's wedding cake

of Gabriel's Horn for higher-dimensional rotational hypersurfaces is briefly discussed (see also Example 3 later in this article).

Here, we introduce more general objects, called *spherical arrays*, that naturally generalize surfaces of revolution and contain hypersurfaces of revolution as a proper subclass. The rich symmetry of these objects can be exploited to develop their volume formulas with relative ease. It is our hope that the explicit nature of these calculations will stimulate interest in differential geometry for those students who have not seen calculus in higher-dimensional Euclidean spaces. An immediate consequence of our study is the development of new examples of hypersurfaces that exhibit properties reminiscent of those for which Gabriel's Horn became a *cause célèbre*.

Generalizing surfaces of revolution

A *surface of revolution*, as typically studied in the first year of a calculus course, is a surface H in $\mathbb{R}^3$ such that the cross-sections orthogonal to a particular line, which we label the x-axis and refer to as the *axis of rotation*, are circles centered on the x-axis, as illustrated in FIGURE 3. There, the notation $S^1(f(x))$ refers to a cross-sectional circle of radius $f(x)$, where $f : \Omega \to [0, \infty)$ is a C^1 function, which we call the *profile function* of H, and $\Omega \subset \mathbb{R}$ is the domain of f.

Algebraically, we write $H = \left\{(x_1, x_2, x) \in \mathbb{R}^2 \times \Omega \subset \mathbb{R}^3 : x_1^2 + x_2^2 = [f(x)]^2\right\}$. Note that if we fix a specific $x = x_0$, the equation reduces to $x_1^2 + x_2^2 = [f(x_0)]^2$, which represents a circle centered about the x-axis at x_0, with radius $f(x_0)$. Intuitively, we may visualize H as a collection of circles stacked alongside one another, with radii varying smoothly according to the value of the profile function. We stress that in this definition, we are decomposing $\mathbb{R}^3$ into two parts: one domainal direction and two rotational directions.

A natural abstraction to 4-dimensional space only requires replacing the cross-sectional circles, which are just 1-dimensional spheres, with cross-sectional 2-dimensional spheres. In particular, we may define a *hypersurface of revolution* H in $\mathbb{R}^4$ algebraically as $H = \left\{(x_1, x_2, x_3, x) \in \mathbb{R}^3 \times \Omega \subset \mathbb{R}^4 : x_1^2 + x_2^2 + x_3^2 = [f(x)]^2\right\}$, where f represents the profile function of H. The intuitive definition is unchanged: A hypersurface of revolution may be thought of as a collection of spheres, stacked alongside one another, with radius varying according to the value of the profile function. And as above, this definition requires a decomposition of $\mathbb{R}^4$ into one domainal dimension and three rotational directions.

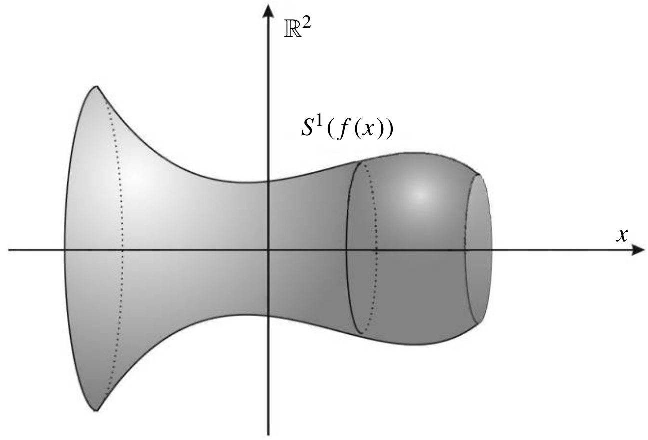


Figure 3 A surface of revolution

But suppose we were to decompose $\mathbb{R}^4$ differently? For example, we may decide to dedicate two dimensions of the ambient $\mathbb{R}^4$ for the domain of the profile function, reserving the remaining two dimensions for rotation. This is formalized in the following definition.

DEFINITION. Given a domain $\Omega \subset \mathbb{R}^2$ and a C^1 profile function $f : \Omega \to [0, \infty)$, we define the *spherical array* $\mathcal{S}_2^3$ *generated by* f to be

$$\mathcal{S}_2^3 = \left\{(x_1, x_2, y_1, y_2) \in \mathbb{R}^2 \times \Omega : x_1^2 + x_2^2 = [f(y_1, y_2)]^2\right\}, \tag{1}$$

where the subscript represents the number of rotational dimensions and the superscript represents the dimension of the spherical array. Note that a surface of revolution would then be an $\mathcal{S}_2^2$ spherical array.

As above, equation (1) is the algebraic statement that for each fixed $y = (y_1, y_2) \in \Omega$, a cross-section of $\mathcal{S}_2^3$ taken perpendicular to Ω is a circle of radius $f(y_1, y_2)$.

Considering all profile functions of two variables would be an overbroad investigation, so we focus on those functions dependent on the norm of $y = (y_1, y_2) \in \Omega \subset \mathbb{R}^2$. Geometrically, the level sets of any such function are (unions of) circles centered at the origin of $\mathbb{R}^2$, and so it is natural to assume that the domain Ω is a ball centered at the origin. We refer to the resulting spherical arrays as *radially symmetric*. In addition to the rotational symmetry exhibited by all spherical arrays, observe that radially symmetric spherical arrays maintain additional symmetry in the domainal directions. See FIGURE 4, where the y_1y_2-plane represents the domainal $\mathbb{R}^2$, and the vertical axis represents the rotational $\mathbb{R}^2$. Note that in FIGURE 4, the point labeled $S^1(f(\|y\|))$ on $\mathcal{S}_2^3$ is a circle.

We remark that, mathematically, spherical arrays appear in a number of guises. If the profile function does not take on the value 0, a spherical array may be viewed topologically as a trivial sphere bundle with base space Ω. Geometrically, such an array has the additional structure of a warped product. Finally, radially symmetric spherical arrays can be studied from the perspective of their isometry groups, which necessarily contain a copy of $O(2) \times O(2)$, where $O(m)$ represents the group of $m \times m$ orthogonal matrices. The first copy of $O(2)$ arises from the two rotational dimensions, while the second copy of $O(2)$ corresponds to the two domainal directions.

In what follows, we assume that all spherical arrays are radially symmetric. In particular, the notation $\mathcal{S}_2^3$ always represents a radially symmetric spherical array.

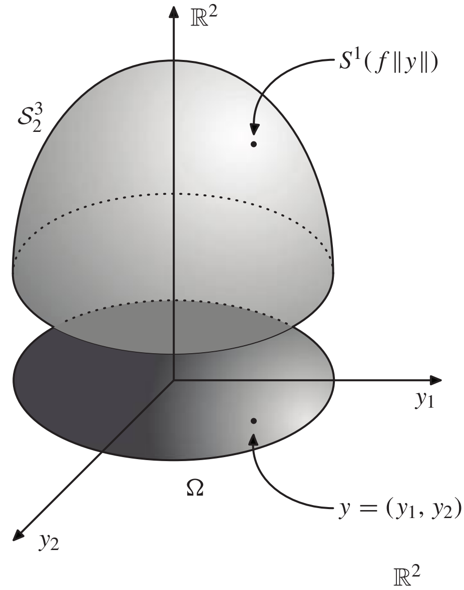


Figure 4 A radially symmetric spherical array in $\mathbb{R}^4$

Volume formula for $\mathcal{S}_2^3$

In the final weeks of a multivariable calculus course, students are required to perform calculations involving parametrized curves and surfaces. The techniques for computing the arc length of a parametrized curve or the surface area of a parametrized surface follow the same recipe: After finding a suitable parametrization, the arc length (1-volume) or surface area (2-volume) is found by integrating some quantity, dependent in a natural way on the parametrization and its first partial derivatives, over the domain of the parametrization. This is illustrated in the following example.

EXAMPLE 1. If $\gamma : [a, b] \to \mathbb{R}^3 : t \mapsto (x(t), y(t), z(t))$ is a parametrized space curve, the *arc length* of $\gamma(t)$ is $\int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\, dt$.

Similar techniques can be used to compute the k-volume of any smooth k-dimensional object H in $\mathbb{R}^{n+1}$. Let σ be a parametrization of H and $\sigma_1, \ldots, \sigma_k$ represent the partial derivatives of σ with respect to the k coordinate directions. Then each σ_i is a vector in $\mathbb{R}^{n+1}$, and we can use these vectors to populate a $k \times k$ matrix as follows:

$$g = \begin{pmatrix} \langle\sigma_1, \sigma_1\rangle & \langle\sigma_1, \sigma_2\rangle & \cdots & \langle\sigma_1, \sigma_k\rangle \\ \langle\sigma_2, \sigma_1\rangle & \langle\sigma_2, \sigma_2\rangle & \cdots & \langle\sigma_2, \sigma_k\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle\sigma_k, \sigma_1\rangle & \langle\sigma_k, \sigma_2\rangle & \cdots & \langle\sigma_k, \sigma_k\rangle \end{pmatrix}, \tag{2}$$

where the notation $\langle\cdot, \cdot\rangle$ represents the dot product in $\mathbb{R}^{n+1}$. The k-volume of H is then given by integrating $\sqrt{|\det(g)|}$ over the domain of the parametrization.

The quantity $\sqrt{|\det(g)|}$ has a nice geometric interpretation. Given a collection of vectors $v_1, \ldots, v_m$ in $\mathbb{R}^{n+1}$, the matrix formed by taking pairwise dot products of these vectors, as in (2), is called the *Gram matrix* (or *Gramian*) of $v_1, \ldots, v_m$. Then, $\sqrt{|\det(g)|}$ represents the m-volume of the m-dimensional parallelepiped spanned by the vectors $v_1, \ldots, v_m$. Intuitively, the volume of an object H parametrized by σ is given by summing, over the domain of the parametrization, the volumes of the infinitesimal parallelepipeds formed by the partial derivatives of σ.

In differential geometry, the matrix in (2) is called the *first fundamental form* and appears in all distance and volume related calculations. The reader may wish to consult [**3**] or [**13**] for more information.

EXAMPLE 2. Given a vector $v = (a, b) \in \mathbb{R}^2$, the Gramian of v is the 1×1 matrix with entry $a^2 + b^2$, so the 1-volume of the parallelepiped (the length of the vector v) is given by $\sqrt{|\det(g)|} = \sqrt{a^2 + b^2}$, which is just the statement of the Pythagorean Theorem! Repeating for two vectors v_1, v_2 in $\mathbb{R}^2$ gives the area of the parallelogram spanned by v_1 and v_2. This extends to a set of vectors $v_1, \dots, v_m$ in $\mathbb{R}^n$ to yield m-volume in n-dimensional space. See the excellent article by Porter [**14**] for more detail on this generalization of the Pythagorean Theorem.

EXERCISE. The procedure described above can be used to verify the formulas for arc length and surface area as presented in a multivariable calculus course. That is, the formula in Example 1 may be derived by computing the quantity $\sqrt{|\det(g)|}$, where $g = (\langle \gamma'(t), \gamma'(t) \rangle)$ is the Gram matrix induced by the parametrization γ. Next, given a subset $U \subset \mathbb{R}^2$, let $\sigma : U \to \mathbb{R}^3 : (u, v) \mapsto \sigma(u, v)$ be a smooth parametrization of a surface S. Then the surface area of S is given by the formula $A(S) = \int_U |\sigma_u \times \sigma_v|\, dU$ (see [**16**], p. 1076). Check that if

$$g = \begin{pmatrix} \langle \sigma_u, \sigma_u \rangle & \langle \sigma_u, \sigma_v \rangle \\ \langle \sigma_v, \sigma_u \rangle & \langle \sigma_v, \sigma_v \rangle \end{pmatrix}$$

is the Gram matrix of the parametrization σ, then $\sqrt{|\det(g)|}$ agrees with the magnitude of the cross product of σ_u and σ_v.

The volume of a radially symmetric spherical array $\mathcal{S}_2^3$ may be computed according to the following three-step process:

1. Find a parametrization σ of $\mathcal{S}_2^3$,
2. Compute the matrix g corresponding to σ,
3. Integrate the quantity $\sqrt{|\det(g)|}$ over the domain of σ.

Theoretically, this three-step program may be used to compute the k-volume of any smooth k-dimensional object in $\mathbb{R}^{n+1}$. In practice, the most common obstacle to carrying out this program occurs at the first step, since finding a parametrization of the target object may be difficult. However, the rich symmetry of a radially symmetric spherical array can be used to find a suitable parametrization. In particular, two sets of polar coordinates can be used: one set for the domain Ω (since it is assumed to be a round ball), and one set to parametrize the cross-sectional circles. If Ω is a closed ball of radius $R \in (0, \infty]$, (where $R = \infty$ corresponds to the case $\Omega = \mathbb{R}^2$), then a parametrization σ of $\mathcal{S}_2^3$ is given by

$$\sigma(\phi, r, \theta) = (f(r)\cos\phi,\ f(r)\sin\phi,\ r\cos\theta,\ r\sin\theta), \tag{3}$$
$$\text{where } 0 \le r \le R \text{ and } 0 \le \phi, \theta \le 2\pi.$$

Indeed, one may check that (3) is equivalent to (1). Note that the final two coordinates, r and θ, parametrize the ball Ω, and for fixed r and θ, the first coordinate, ϕ, parametrizes the cross-sectional circles of radius $f(r)$.

The second step is to compute the matrix g. We first compute the partial derivatives

$$\begin{aligned}\sigma_\phi &= (-f(r)\sin\phi,\ f(r)\cos\phi,\ 0,\ 0),\\ \sigma_r &= (f'(r)\cos\phi,\ f'(r)\sin\phi,\ \cos\theta,\ \sin\theta),\\ \sigma_\theta &= (0,\ 0,\ -r\sin\theta,\ r\cos\theta).\end{aligned}$$

So, we have

$$g = \begin{pmatrix} \langle\sigma_\phi, \sigma_\phi\rangle & \langle\sigma_\phi, \sigma_r\rangle & \langle\sigma_\phi, \sigma_\theta\rangle \\ \langle\sigma_r, \sigma_\phi\rangle & \langle\sigma_r, \sigma_r\rangle & \langle\sigma_r, \sigma_\theta\rangle \\ \langle\sigma_\theta, \sigma_\phi\rangle & \langle\sigma_\theta, \sigma_r\rangle & \langle\sigma_\theta, \sigma_\theta\rangle \end{pmatrix} = \begin{pmatrix} [f(r)]^2 & 0 & 0 \\ 0 & 1+[f'(r)]^2 & 0 \\ 0 & 0 & r^2 \end{pmatrix}.$$

We are left with the integration of the quantity $\sqrt{|\det(g)|}$ over the domain of the parametrization.

$$\begin{aligned}\mathrm{Vol}(\mathcal{S}_2^3) &= \int_0^{2\pi}\int_0^R\int_0^{2\pi} \sqrt{|\det(g)|}\, d\phi\, dr\, d\theta \\ &= \int_0^{2\pi}\int_0^R\int_0^{2\pi} r\cdot f(r)\cdot\sqrt{1+[f'(r)]^2}\, d\phi\, dr\, d\theta \\ &= 4\pi^2\int_0^R r\cdot f(r)\cdot\sqrt{1+[f'(r)]^2}\, dr. \qquad (4)\end{aligned}$$

This completes the three-step process, and we are rewarded with the volume of a radially symmetric spherical array.

Volume formula for $\mathcal{E}_2^4$

This process can be replicated to find the volume of the solid $\mathcal{E}_2^4$ enclosed by $\mathcal{S}_2^3$. We formally define this object by

$$\mathcal{E}_2^4 = \left\{(x_1, x_2, y_1, y_2) \in \mathbb{R}^2 \times \Omega : x_1^2 + x_2^2 \le [f(y_1, y_2)]^2\right\}.$$

Assuming again that f is radially symmetric, we can parametrize $\mathcal{E}_2^4$ by modifying the parametrization of $\mathcal{S}_2^3$, in the sense that we need only change the first two coordinates so that they parametrize cross-sectional solid disks of radius $f(r)$, instead of circles of radius $f(r)$. Thus, a parametrization σ of $\mathcal{E}_2^4$ is given by

$$\sigma(\rho, \phi, r, \theta) = (\rho\cos\phi,\ \rho\sin\phi,\ r\cos\theta,\ r\sin\theta),$$
$$\text{where } 0 \le \rho \le f(r),\ 0 \le r \le R,\ \text{and } 0 \le \phi, \theta \le 2\pi.$$

We then have

$$\begin{aligned}\sigma_\rho &= (\cos\phi, \sin\phi, 0, 0),\\ \sigma_\phi &= (-\rho\sin\phi, \rho\cos\phi, 0, 0),\\ \sigma_r &= (0, 0, \cos\theta, \sin\theta),\\ \sigma_\theta &= (0, 0, -r\sin\theta, r\cos\theta),\end{aligned}$$

so that

$$g = \begin{pmatrix} \langle\sigma_\rho,\sigma_\rho\rangle & \langle\sigma_\rho,\sigma_\phi\rangle & \langle\sigma_\rho,\sigma_r\rangle & \langle\sigma_\rho,\sigma_\theta\rangle \\ \langle\sigma_\phi,\sigma_\rho\rangle & \langle\sigma_\phi,\sigma_\phi\rangle & \langle\sigma_\phi,\sigma_r\rangle & \langle\sigma_\phi,\sigma_\theta\rangle \\ \langle\sigma_r,\sigma_\rho\rangle & \langle\sigma_r,\sigma_\phi\rangle & \langle\sigma_r,\sigma_r\rangle & \langle\sigma_r,\sigma_\theta\rangle \\ \langle\sigma_\theta,\sigma_\rho\rangle & \langle\sigma_\theta,\sigma_\phi\rangle & \langle\sigma_\theta,\sigma_r\rangle & \langle\sigma_\theta,\sigma_\theta\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \rho^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & r^2 \end{pmatrix}.$$

It follows that the volume of $\mathcal{E}_2^4$ is given by

$$\begin{aligned} \mathrm{Vol}(\mathcal{E}_2^4) &= \int_0^{2\pi}\int_0^R\int_0^{2\pi}\int_0^{f(r)} \sqrt{|\det(g)|}\, d\rho\, d\phi\, dr\, d\theta \\ &= \int_0^{2\pi}\int_0^R\int_0^{2\pi}\int_0^{f(r)} \rho\cdot r\, d\rho\, d\phi\, dr\, d\theta \\ &= 2\pi^2\int_0^R r\cdot[f(r)]^2\, dr. \end{aligned} \tag{5}$$

Volume formulas for $\mathcal{S}_k^n$

The spherical arrays discussed in the previous section arise from a specific decomposition of $\mathbb{R}^4$, but the setup generalizes neatly to higher dimensions and other decompositions of the dimension of the ambient space. Given $n \geq 2$, choose an integer k $(1 < k \leq n)$ to represent the number of rotational dimensions. Now, given $\Omega \subset \mathbb{R}^{n-k+1}$ and a profile function $f : \Omega \to [0,\infty)$, define a spherical array $\mathcal{S}_k^n$ implicitly as

$$\mathcal{S}_k^n = \big\{(x_1,\ldots,x_k,y_1,\ldots,y_{n-k+1}) \in \mathbb{R}^k \times \Omega \subset \mathbb{R}^{n+1} : \\ x_1^2+\cdots+x_k^2 = [f(y_1,\ldots,y_{n-k+1})]^2\big\}.$$

The spherical arrays $\mathcal{S}_2^3$ discussed above occur when $n = 3$ and $k = 2$. The special case $k = n$ yields an n-dimensional hypersurface of revolution.

We proceed with a heuristic argument for the volume formulas of a generic, radially symmetric spherical array. Let $a_{m-1}(\delta)$ represent the $(m-1)$-volume of the $(m-1)$-dimensional sphere of radius δ and $v_m(\delta)$ represent the m-volume of the m-dimensional solid ball of radius δ. The exact values of $a_{m-1}(\delta)$ and $v_m(\delta)$ are not needed, but we do make use of the fact that there exists a positive constant C_m such that

$$a_{m-1}(\delta) = mC_m\delta^{m-1} \qquad \text{and} \qquad v_m(\delta) = C_m\delta^m. \tag{6}$$

When $m = 2$, $a_1(\delta) = 2\pi\delta$ is the circumference of a circle of radius δ, and $v_2(\delta) = \pi\delta^2$ represents the area of a disk of radius δ. Therefore, we can rewrite (4) and (5) as

$$\begin{aligned} \mathrm{Vol}(\mathcal{S}_2^3) &= \int_0^R 2\pi r\cdot 2\pi f(r)\cdot\sqrt{1+[f'(r)]^2}\, dr \\ &= \int_0^R a_1(r)\cdot a_1(f(r))\cdot\sqrt{1+[f'(r)]^2}\, dr, \quad \text{and} \\ \mathrm{Vol}(\mathcal{E}_2^4) &= \int_0^R 2\pi r\cdot\pi[f(r)]^2\, dr \\ &= \int_0^R a_1(r)\cdot v_2(f(r))\, dr. \end{aligned}$$

The formulas above generalize neatly to higher-dimensional spherical arrays.

THEOREM 1. *Given $n \geq 2$ and $1 < k \leq n$, let Ω be the $(n-k+1)$-dimensional solid ball of radius R. If $f : \Omega \to [0, \infty)$ is the profile function of a radially symmetric spherical array $\mathcal{S}_k^n$, then the volume of $\mathcal{S}_k^n$ and its enclosed solid $\mathcal{E}_k^{n+1}$ are, respectively, given by*

$$\begin{aligned}\mathrm{Vol}(\mathcal{S}_k^n) &= \int_0^R a_{n-k}(r) \cdot a_{k-1}(f(r)) \cdot \sqrt{1+[f'(r)]^2}\, dr \\ &= a_{n-k}(1) \cdot a_{k-1}(1) \int_0^R r^{n-k} \cdot [f(r)]^{k-1} \cdot \sqrt{1+[f'(r)]^2}\, dr, \qquad (7)\end{aligned}$$

$$\begin{aligned}\mathrm{Vol}(\mathcal{E}_k^{n+1}) &= \int_0^R a_{n-k}(r) \cdot v_k(f(r))\, dr \\ &= a_{n-k}(1) \cdot v_k(1) \int_0^R r^{n-k} \cdot [f(r)]^k\, dr. \qquad (8)\end{aligned}$$

Observe that in each case, the second equality follows from (6). For the first equalities, the same technique as above can be used. When $n = 5$ and $k = 3$, a radially symmetric spherical array can be parametrized with two sets of spherical coordinates as follows:

$$\begin{aligned}\sigma(\phi_1, \phi_2, r, \theta_1, \theta_2) = (&f(r)\sin\phi_1\sin\phi_2,\ f(r)\sin\phi_1\cos\phi_2,\ f(r)\cos\phi_1, \\ &r\sin\theta_1\sin\theta_2, r\sin\theta_1\cos\theta_2, r\cos\theta_1), \\ \text{where } 0 \leq r \leq R,\ &0 \leq \phi_1, \theta_1 \leq \pi,\ 0 \leq \phi_2, \theta_2 \leq 2\pi.\end{aligned}$$

One may check, after computing the partial derivatives, that the matrix g is given by

$$g = \begin{pmatrix} [f(r)]^2 & 0 & 0 & 0 & 0 \\ 0 & \sin^2\phi_1\,[f(r)]^2 & 0 & 0 & 0 \\ 0 & 0 & 1+[f'(r)]^2 & 0 & 0 \\ 0 & 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & 0 & \sin^2\theta_1\, r^2 \end{pmatrix}.$$

The formula for $\mathrm{Vol}(\mathcal{S}_3^5)$ now follows from setting up the volume integral and applying equation (1.10) from **[1]**. The volume of the solid array may be computed similarly. In higher dimensions, two sets of *hyperspherical coordinates* [**18**] may be used.

Generalizing Gabriel's Horn

We now turn our attention to building n-dimensional spherical arrays with the Horn Property. Armed with equations (7) and (8), and the knowledge that $\int_1^\infty r^{-p}\, dr$ converges for $p > 1$, this is a straightforward task.

THEOREM 2. *Let $\Omega = \{y = (y_1, \ldots, y_{n-k+1}) \in \mathbb{R}^{n-k+1} : 1 \leq \|y\| = r\}$ and let $\mathcal{G}_k^n$ be the radially symmetric spherical array defined by*

$$\mathcal{G}_k^n = \Big\{(x_1, \ldots, x_k, y_1, \ldots, y_{n-k+1}) \in \mathbb{R}^k \times \Omega \subset \mathbb{R}^{n+1} : \\ x_1^2 + \cdots + x_k^2 = \left(y_1^2 + \cdots + y_{n-k+1}^2\right)^{-p}\Big\}.$$

Then $\mathcal{G}_k^n$ exhibits the Horn Property when $\frac{n-k+1}{k} < p \leq \frac{n-k+1}{k-1}$.

Proof. The hypersurface $\mathcal{G}_k^n$ is the radially symmetric spherical array generated by the profile function $f(r) = r^{-p}$ on the domain Ω. From (8), the volume enclosed by $\mathcal{G}_k^n$ is proportional to

$$\int_1^\infty r^{n-k} \cdot [f(r)]^k \, dr = \int_1^\infty r^{n-k} \cdot r^{-pk} \, dr = \int_1^\infty r^{n-k-pk} \, dr,$$

which is finite when $p > \frac{n-k+1}{k}$. On the other hand, we have

$$\begin{aligned}\int_1^\infty r^{n-k} \cdot [f(r)]^{k-1} \cdot \sqrt{1 + [f'(r)]^2} \, dr &\geq \int_1^\infty r^{n-k} \cdot [f(r)]^{k-1} \, dr \\ &= \int_1^\infty r^{n-k} \cdot r^{-p(k-1)} \, dr \\ &= \int_1^\infty r^{n-k-p(k-1)} \, dr,\end{aligned}$$

so by (7), the volume of $\mathcal{G}_k^n$ is infinite when $p \leq \frac{n-k+1}{k-1}$. ■

EXAMPLE 3. If $f(x) = x^{-p}$ is used to generate an n-dimensional hypersurface of revolution H in $\mathbb{R}^{n+1}$, then H has the Horn Property precisely when $\frac{1}{n} < p \leq \frac{1}{n-1}$ (see [**1**]).

More objects

So far, we have restricted our attention to hypersurfaces with spherical cross-sections. Can other objects be used as cross-sections? Certainly! In particular, nontrivial hypersurfaces arise when the cross-sectional objects are other than topological spheres. So long as the chosen object admits a reasonable parametrization, the volume formulas associated with the resulting hypersurface will be computable by the method outlined earlier. For example, a *torus* is the surface of revolution generated by revolving a circle about a coplanar, non-incident axis. If the circle has radius R_1 and the center of the circle is distance R_2 from the axis, the torus is said to have inner radius R_1 and outer radius R_2 (see FIGURE 5).

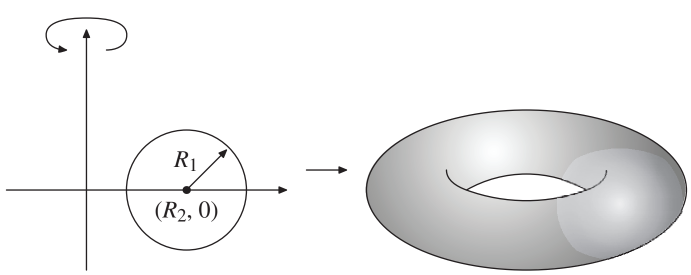


Figure 5 A torus as a surface of revolution in $\mathbb{R}^3$

Defining a hypersurface with toroidal cross-sections first requires a parametrization of the torus itself. Letting R_1 and R_2 be as above, the standard parametrization of a torus embedded in $\mathbb{R}^3$ is given by

$$\sigma(\phi_1, \phi_2) = (\cos\phi_1(R_2 + R_1\cos\phi_2),\ \sin\phi_1(R_2 + R_1\cos\phi_2),\ R_1\sin\phi_2),$$
$$\text{where } 0 \leq \phi_1, \phi_2 < 2\pi.$$

As before, assume that $\Omega \subset \mathbb{R}^2$ is a ball of radius R centered at the origin and that $f : \Omega \to (0, \infty)$ is a radially symmetric profile function. We then define a 4-dimensional *toroidal array* in $\mathbb{R}^5$ so that for each $y \in \Omega$, the cross-section in the three directions orthogonal to Ω is a torus with inner radius $f(r)$, $r = \|y\|$, and outer radius 4. In this case, to avoid self-intersection, we insist that $f(r) < 2$. Whence, a parametrization σ_1 for a toroidal array $\mathcal{T}$ is given by

$$\begin{aligned}\sigma_1(\phi_1, \phi_2, r, \theta) = (&\cos\phi_1(4 + f(r)\cos\phi_2),\ \sin\phi_1(4 + f(r)\cos\phi_2),\\ &f(r)\sin\phi_2,\ r\cos\theta,\ r\sin\theta),\\ &\text{where } 0 \le r \le R \text{ and } 0 \le \phi_1, \phi_2, \theta < 2\pi.\end{aligned}$$

As in the parametrization of a spherical array, note that the final two coordinates, r and θ, parametrize the ball Ω, and for fixed r and θ, the first two coordinates, ϕ_1 and ϕ_2, parametrize the cross-sectional torus with outer radius 4 and inner radius $f(r)$.

Similarly, a parametrization σ_2 for the solid T contained inside $\mathcal{T}$ is given by

$$\begin{aligned}\sigma_2(\rho, \phi_1, \phi_2, r, \theta) = (&\cos\phi_1(4 + \rho\cos\phi_2),\ \sin\phi_1(4 + \rho\cos\phi_2),\\ &\rho\sin\phi_2,\ r\cos\theta,\ r\sin\theta),\\ &\text{where } 0 \le r \le R,\ 0 \le \rho \le f(r), \text{ and } 0 \le \phi_1, \phi_2, \theta < 2\pi.\end{aligned}$$

EXERCISE. Show that the volume and enclosed volume of a radially symmetric toroidal array with outer radius 4 are given by

$$\begin{aligned}\mathrm{Vol}(\mathcal{T}) &= 32\pi^3 \int_0^R r \cdot f(r) \cdot \sqrt{1 + [f'(r)]^2}\, dr\\ &= 8\pi \cdot a_1(1) \cdot a_1(1) \int_0^R r \cdot f(r) \cdot \sqrt{1 + [f'(r)]^2}\, dr\\ &= 8\pi \cdot \mathrm{Vol}(\mathcal{S}_2^3),\end{aligned}$$

and

$$\begin{aligned}\mathrm{Vol}(T) &= 16\pi^3 \int_0^R r \cdot [f(r)]^2\, dr\\ &= 8\pi \cdot a_1(1) \cdot v_2(1) \int_0^R r \cdot f(r)^2\, dr\\ &= 8\pi \cdot \mathrm{Vol}(\mathcal{E}_2^4).\end{aligned}$$

It follows from the above formulas that if $\Omega = \left\{y = (y_1, y_2) \in \mathbb{R}^2 : \|y\| \ge 1\right\}$ with $f(r) = r^{-p}$ for $1 < p \le 2$, the resulting toroidal array has the Horn Property. Of course, the existence of topologically nontrivial versions of Gabriel's Horn is not surprising. For example, we may picture removing a smaller solid horn-shaped object from the interior of the ordinary, solid version of Gabriel's Horn. The cross-sections of the remaining object are annular disks centered along the axis of revolution, and the resulting 3-volume is decreased while the surface area of the object is increased—so the Horn Property is maintained. This type of construction is viable in any dimension. Explicitly, we may build such an object by generating a hypersolid of revolution from the region between the curves $g(x) = x^{-1}$ and $h(x) = x^{-2}$.

The interested reader is encouraged to consider an array with different cross-sections, or possibly a toroidal array where the outer radius $F(r)$ is not a constant. This collection of examples should illustrate that the novelty of new hypersurfaces with the Horn Property is limited only by the creativity of the reader.

Afterward

Gabriel's Horn is an unbounded surface of revolution with finite volume. In [**11**], a famous function from classical analysis is modified to develop a compact surface of revolution that has the Horn Property. However, the profile curve for this example is only differentiable and not of C^1 smoothness. Indeed, a compact surface of revolution (with profile function f) exhibiting the Horn Property cannot have C^1 smoothness, since this would imply that both f and f' are bounded, and so too, therefore, is the integral that defines the surface area. Similarly, since spherical arrays are dependent on a function of one variable, they cannot be both compact and C^1. However, spherical arrays that are both compact and smooth (in fact, analytic) have been studied in another context. In [**5**], the authors use these hypersurfaces to produce examples of objects that satisfy a higher-dimensional analogue of a classical property of the sphere whose discovery dates back to Archimedes—that the surface area of a zone of the sphere lying between two parallel planes depends only on the distance between the planes [**4, 6, 15**]. Also in [**5**], an alternate method for proving Theorem 1 is given.

Acknowledgment The authors would like to thank Jeff Dodd and David L. Johnson for reading numerous drafts of this paper and for making numerous helpful suggestions.

REFERENCES

1. D. Aberra and K. Agrawal, Surfaces of revolution in n dimensions, *International Journal of Mathematical Education in Science and Technology* **38** (2007) 715–727. http://dx.doi.org/10.1080/00207390701359388
2. M. Carroll, S. Dougherty, and D. Perkins, Indivisibles, infinitesimals, and a tale of seventeenth-century mathematics, *Math. Mag.* **86** (2013) 239–254. http://dx.doi.org/10.4169/math.mag.86.4.239
3. M. do Carmo, *Differential Geometry of Curves and Surfaces*, Prentice-Hall, Englewood Cliffs, NJ, 1976.
4. V. Coll and J. Dodd, Generalizing the equal area zones property of the sphere, *J. Geom.* **90** (2008) 47–55. http://dx.doi.org/10.1007/s00022-008-2015-2
5. V. Coll, J. Dodd, and M. Harrison, Archimedean hypersurfaces (forthcoming).
6. ———, On the smoothness of the equizonal ovaloids, *J. Geom.* **103** (2012) 409–416. http://dx.doi.org/10.1007/s00022-012-0135-1
7. B. Eisenberg, Surfaces of revolution in four dimensions, *Math. Mag.* **77** (2004) 379–386. http://dx.doi.org/10.2307/3219204
8. J. Fleron, Gabriel's wedding cake, *College Math. J.* **30** (1999) 35–38. http://dx.doi.org/10.2307/2687201
9. T. Hobbes, *The English Works of Thomas Hobbes, Volume 7*, Longman, Brown, Green, and Longmans, London, 1845, 445.
10. C. Huygens, *Oeuvres Complètes, Volume 2*, Martinus Nijhoff, La Haye, Netherlands, 1891, 168.
11. J. Krenicky and J. Rychtář, Note on Gabriel's horn, *Electronic Journal of Undergraduate Mathematics* **11** (2006) 4–7. http://math.furman.edu/~mwoodard/fuejum/content/2006/paper2_2006.pdf
12. P. Mancosu and E. Vailati, Torricelli's infinitely long solid and its philosophical reception in the seventeenth century, *Isis* **82** (1991) 50–70. http://dx.doi.org/10.1086/355637
13. R. Millman and G. Parker, *Elements of Differential Geometry*, Prentice-Hall, Upper Saddle River, NJ, 1997.
14. G. Porter, k-Volume in $\mathbb{R}^n$ and the Generalized Pythagorean Theorem, *Amer. Math. Monthly* **103** (1996) 252–256. http://dx.doi.org/10.2307/2975377
15. B. Richmond and T. Richmond, The equal area zones property, *Amer. Math. Monthly* **100** (1993) 475–477. http://dx.doi.org/10.2307/2324302
16. J. Stewart, *Calculus: Early Transcendentals*, 6th ed., Thomson, Belmont, CA, 2008.
17. H. von Koch, Sur une courbe continue sans tangente, obtenue par une construction géométrique élémentaire, *Arkiv for Matematik* **1** (1904) 681–704.
18. n-sphere, *Wikipedia*, available at http://en.wikipedia.org/wiki/N-sphere.

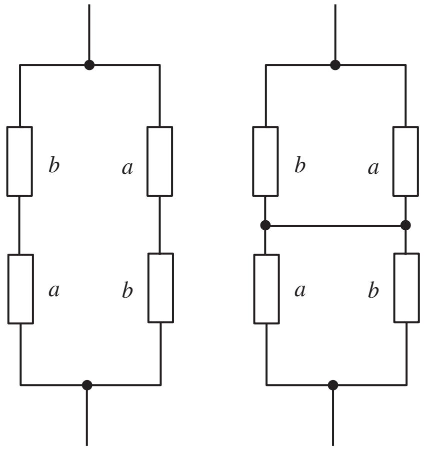
Summary We generalize hypersurfaces of revolution by allowing the profile curve to be dependent on more than one variable. We call these more general objects spherical arrays and we use them to introduce the reader to volume calculations, which are normally reserved for a more advanced course in differential geometry. The rich symmetry of the spherical arrays allow us to build objects with properties reminiscent of those for which Gabriel's Horn became a *cause célèbre*.

VINCENT E. COLL, JR. received a B.S. from Loyola University (New Orleans), an M.S. from Texas A&M University, and a Ph.D. in algebraic deformation theory from the University of Pennsylvania in 1990 under the direction of Murray Gerstenhaber. Since 2008, he has been on the faculty at Lehigh University. Aside from the geometry of hypersurfaces of revolution, his research interests include cohomological algebra and the study of Frobenius seaweed Lie algebras. His outside interests include the martial arts and ice hockey—but not at the same time.

MICHAEL HARRISON received B.S. degrees in mathematics and statistics from Pennsylvania State University in 2008 and an M.Sc. in mathematics at King's College London in 2011. He has since returned to Penn State to pursue a Ph.D. in differential topology under the direction of advisers Sergei Tabachnikov and Kris Wysocki. His other interests include baseball and playing the guitar.

Proof Without Words: An Electrical Proof of the AM-HM Inequality

ALFRED WITKOWSKI
Institute of Mathematics and Physics
University of Technology and Life Sciences
Bydgoszcz, Poland
alfred.witkowski@utp.edu.pl



$$R_L = \frac{a+b}{2} \geq \frac{2ab}{a+b} = R_R$$

That is, AM $\geq$ HM; the arithmetic is at least equal to the harmonic mean. The inequality extends to the geometric mean as well: AM $\geq$ GM $\geq$ HM, because $\text{GM}^2 = AM \cdot HM$.

This proof has appeared in at least two books [**1, 2**], both of which explore the rich connections between mathematics and physics.

REFERENCES

1. Mark Levi, *The Mathematical Mechanic: Using Physical Reasoning to Solve Problems,* Princeton University Press, Princeton, NJ, 2009.
2. P. G. Doyle and J. L. Snell, *Random Walks and Electric Networks,* Mathematical Association of America, Washington, DC, 1984.

Summary The AM-HM inequality arises from comparing the resistances in two circuits.

Math. Mag. **87** (2014) 275. doi:10.4169/math.mag.87.4.275.

NOTES

Proving the Reflective Property of an Ellipse

STEPHAN BERENDONK
Mathematisches Institut, Universität Bonn
53115 Bonn, Germany
berendon@math.uni-bonn.de

"How can I construct the tangent at a point P of an ellipse?" I asked a friend of mine.

"Well, what do you already know about ellipses?" he asked me in return.

"Only the definition," I answered.

"Which definition?" he asked.

So I said, "An ellipse is the locus of points whose sum of distances from two given points, called the foci, is equal to a constant λ."

"Taking this definition, which is also called the *focal property* of an ellipse, we can describe the tangent at P as the exterior angle bisector of $\angle F_1PF_2$ (FIGURE 1). This is because the ellipse has a nice reflective property: Light rays emanating from the focus F_1 and hitting the ellipse at P are, taking the tangent as a mirror, reflected to the other focus F_2. This is why F_1 and F_2 are called focal points," my friend explained.

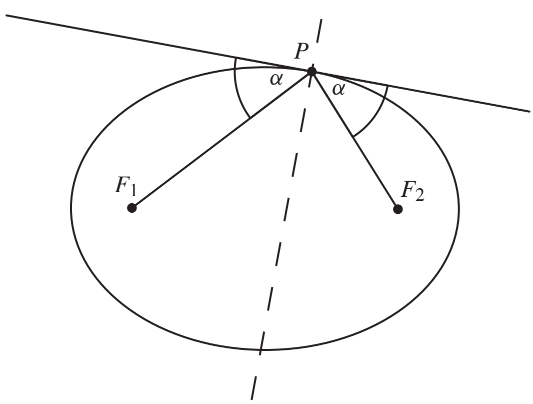


Figure 1

When I asked him to prove that the exterior angle bisector actually yields the tangent, he gave the following argument: Consider a point Q different from P on the exterior angle bisector and let F_2' be the point you get by reflecting F_2 across the line PQ (FIGURE 2). Then we have $|QF_2| = |QF_2'|$ and $|PF_2| = |PF_2'|$. Since PQ is the exterior angle bisector of $\angle F_1PF_2$, the point P lies on the line F_1F_2'. Thus we can use the triangle inequality to get:

$$|F_1Q| + |QF_2| = |F_1Q| + |QF_2'| > |F_1P| + |PF_2'| = |F_1P| + |PF_2| = \lambda.$$

Math. Mag. **87** (2014) 276–279. doi:10.4169/math.mag.87.4.276.

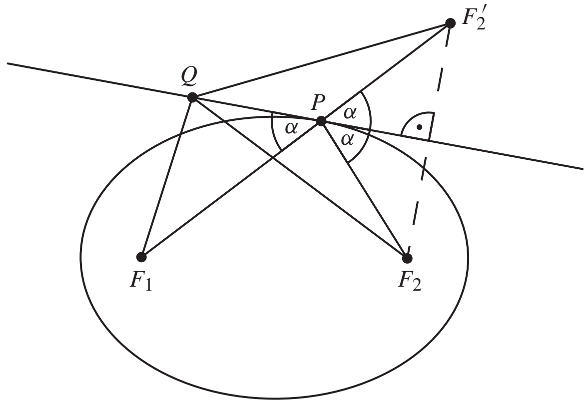


Figure 2

Thus Q lies outside the ellipse. Therefore P is the only point of the exterior angle bisector that lies on the ellipse. Hence, this line must be the tangent.

The last step in his proof confused me. Apparently my friend employed a different notion of tangent than the one I was used to from analysis. To him, a tangent is a line that has only one point in common with the curve. When I scrolled through the literature I realized that this is a common way to define tangents when dealing with ellipses (see [**1**, p. 192], [**2**, p. 58], [**5**, p. 119], [**6**, pp. 2–3], [**7**, p. 200], [**8**, p. 21]). However in Besant's book [**3**], which is possibly the most thorough geometric treatment of the conic sections, I finally found the definition of tangents that I was looking for:

> "If a straight line, drawn through a point P of a curve, meets the curve again in P', and if the straight line be turned round the point P until the point P' approaches indefinitely near to P, the ultimate position of the straight line is the tangent to the curve at P."

Besant, on the other hand, defines the ellipse as the locus of points whose quotient of distances to a line and a point is constant and greater than 1. This leads him also to a different construction of the tangent [**3**, p. 10], which is probably why my friend asked me which definition of an ellipse I had learned. Later on Besant still proves the reflective property of the tangent [**3**, pp. 59–60], but his proof is based on his construction of the tangent, and thus ultimately on the definition of the ellipse that he chose. In any case, his proof does not use the focal property of the ellipse. So I was wondering whether there was a proof of the reflective property based on the focal property and Besant's notion of a tangent. I will now proceed with such a proof.

Let P and P' be points on an ellipse that is given by the foci F_1 and F_2 and the constant λ. Let c be the so-called *directrix circle* with center F_2 and radius λ, and let Q (resp. Q') be the intersection of c with the line F_2P (resp. F_2P') (FIGURE 3). Finally, let S be the intersection of QQ' and PP', and let R be the intersection of $Q'Q$ and the line parallel to F_2Q through P'.

Because $Q'RP'$ and $Q'QF_2$ are similar triangles, we have

$$\frac{P'Q'}{P'R} = \frac{F_2Q'}{F_2Q} = \frac{\lambda}{\lambda} = 1.$$

Thus, by using the focal property we get

$$P'R = P'Q' = \lambda - P'F_2 = P'F_1. \tag{1}$$

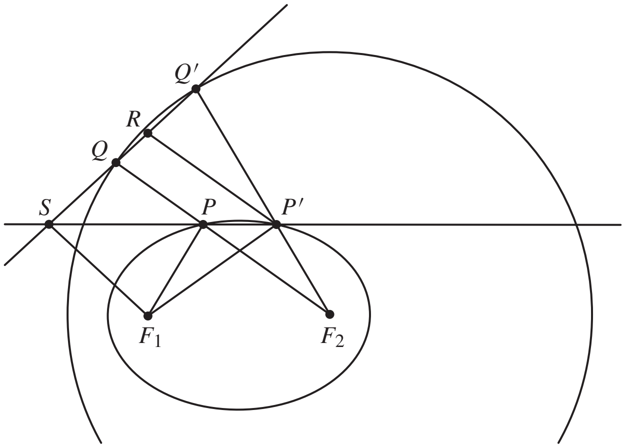


Figure 3

The focal property also gives

$$PQ = \lambda - PF_2 = PF_1. \tag{2}$$

Using equations (1) and (2) together with the similarity of triangles SPQ and $SP'R$ yields

$$\frac{SP}{SP'} = \frac{PQ}{P'R} = \frac{F_1P}{F_1P'}.$$

Hence, the point S divides the side $P'P$ of triangle $F_1P'P$ externally in the ratio of the sides containing $\angle P'F_1P$. Due to the exterior angle bisector theorem (see [**4**, vol. 2, notes to VI.3, pp. 197–198], S lies on the exterior angle bisector of $\angle P'F_1P$.

Now if P' approaches P, the line PP' ultimately becomes the tangent of the ellipse at P and, as the exterior and interior angle bisectors of an angle are perpendicular, S ultimately lies on the perpendicular to F_1P through F_1. At the same time, when P' approaches P, the point Q' moves towards Q so that the line QQ' ultimately turns into the tangent of the circle c at Q. Thus in the limit the point S lies also on the line perpendicular to F_2Q through Q.

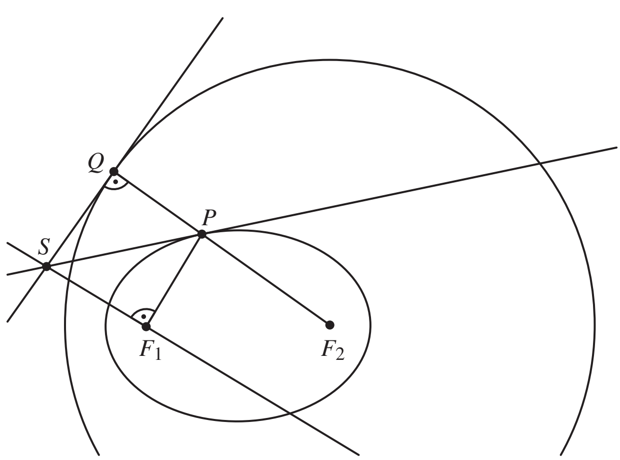


Figure 4

Hence, intersecting the line perpendicular to PF_1 through F_1 with the line perpendicular to F_2Q through Q gives a point S that lies on the tangent of the ellipse at P (FIGURE 4). The triangles SQP and SF_1P are congruent, since they are both right-angled, have the same hypotenuse, and their legs PQ and PF_1 are congruent, by equation (2). Thus,

$$\angle QPS = \angle F_1PS.$$

From this we may conclude that the tangent SP of the ellipse at P is indeed the exterior angle bisector of $\angle F_1PF_2$.

REFERENCES

1. J. M. Aarts, *Plane and Solid Geometry*, Springer, New York, 2008.
2. I. Agricola and T. Friedrich, *Elementargeometrie*, Vieweg+Teubner Verlag, Wiesbaden, Germany, 2011.
3. W. H. Besant, *Conic Sections, Treated Geometrically*, 9th ed., Bell and Sons, London, 1895.
4. T. L. Heath, *The Thirteen Books of Euclid's Elements*, Dover, New York, 1965.
5. A. Hess, *Planimetrie*, Springer-Verlag, Berlin, 1961.
6. D. Hilbert and S. Cohn-Vossen, *Geometry and the Imagination*, Chelsea Publishing, New York, 1990.
7. H. Scheid and W. Schwarz, *Elemente der Geometrie*, Spektrum Akademischer Verlag, Munich, 2007.
8. H. Schupp, *Kegelschnitte*, Franzbecker, Hildesheim, Germany, 2000.

Summary The tangent to an ellipse at a point P can be constructed as the exterior angle bisector of the angle that the point P makes with the two foci of the ellipse. Typically this fact is proved by showing that the exterior angle bisector cannot meet the ellipse in a second point, and therefore must be the tangent. This note gives an alternative proof that is more in line with the notion of tangent as a limit of secants.

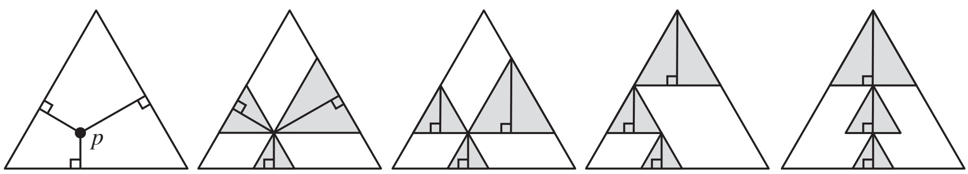
Viviani à la Kawasaki: Take Two

BURKARD POLSTER
Monash University
Victoria 3800, Australia
burkard.polster@monash.edu

Viviani's theorem highlights a surprising property of equilateral triangles:

In an equilateral triangle, the sum of the distances from any interior point p to the three sides of the triangle is equal to the height of the triangle.

Here is a slightly extended version of a very pretty proof without words by Kawasaki [**4**].

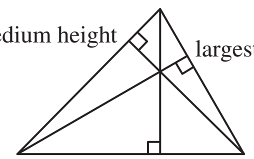


In Harold Jacobs' classic geometry textbook [**3**, pp. 2–6], Viviani's theorem forms the basis for a puzzle. Jacobs imagines a surfer stranded on an island in the shape of an equilateral triangle. (Happens all the time!) Our surfer dude enjoys all of the island's beaches and plans to spend an equal amount of time at each. That suggests that his hut would be most conveniently located so that the sum of the distances to the three beaches is as small as possible. Where would that be?

Of course, Viviani's theorem tells us that it does not matter where the surfer builds his hut. But what if the surfer is stranded on an island of a different shape? We'd like to present you with a natural twist to Kawasaki's argument that provides the answer for all triangles. We start by proving the following:

Let S be the sum of the distances from a point p inside a triangle to its three sides. Then

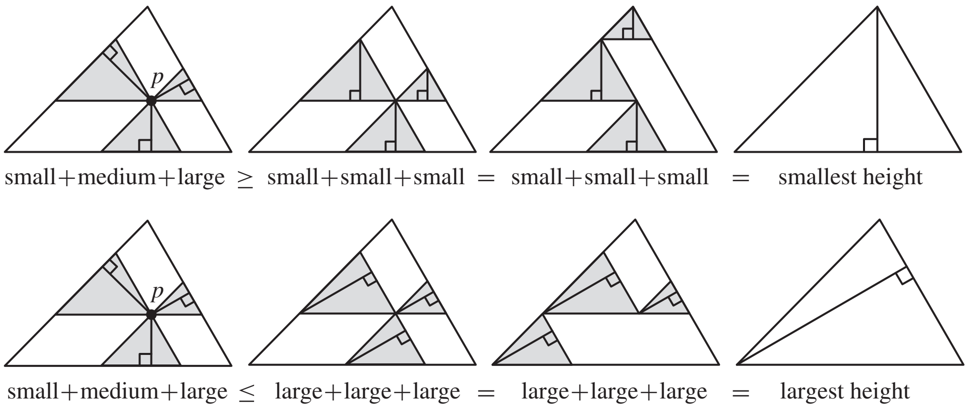
$$\text{the smallest height of the triangle} \leq S \leq \text{the largest height of the triangle.}$$



(For everything we say to work also for obtuse triangles, the *distance to a side of a triangle* has to be interpreted as the *distance to the line that the side is contained in*.)

Math. Mag. **87** (2014) 280–283. doi:10.4169/math.mag.87.4.280.

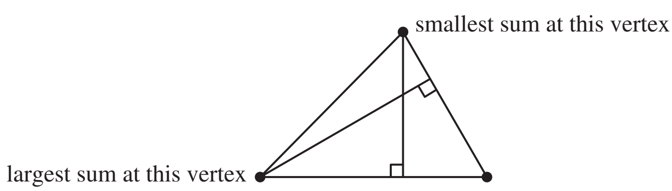
Proof (almost without words).



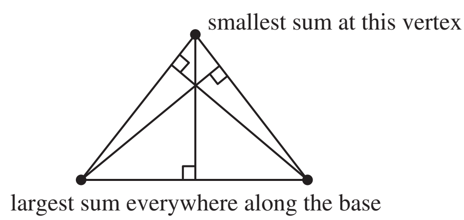
In the case of an equilateral triangle, all three heights are equal. Hence, in this case, our result implies Viviani's theorem. ■

Now, let's have a close look at the diagrams to figure out where the surfer should build his hut on a given triangular island, and where he should definitely not build his hut. In other words, let's figure out at which points of our triangle the sum of the distances to the sides is minimal and maximal. There are three cases to consider.

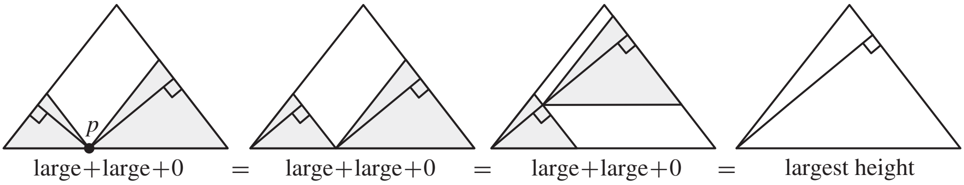
Case 1: Generic triangle in which the three heights are different. In such a triangle the first equality is strict except if the point p is the vertex at the end of the smallest height. As well, the second equality is strict unless the point p is the vertex at the end of the longest height. This means that the sum S takes on its minimal and maximal values at these two vertices.



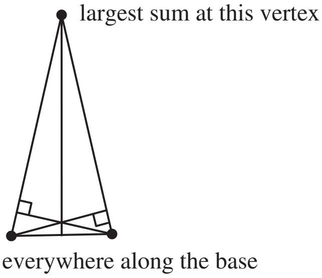
Case 2: Proper isosceles triangle in which the smallest height is the symmetry axis. As in the generic case, we conclude that the vertex at the end of the smallest height corresponds to the smallest sum. The largest sum occurs everywhere along the base of the triangle.



To see the latter, note that there are two longest heights. This means that the second inequality is strict unless the point p is contained in the base of the triangle. Let's redraw the sequence of diagrams with the point p on the longest side to see at a glance that, in this case, the sum of the distances is equal to the longest height; see also [**2**].



Case 3: Proper isosceles triangle in which the largest height is the symmetry axis. As in the previous case, we conclude that the minimal and maximal sums occur at the points highlighted in the following diagram.



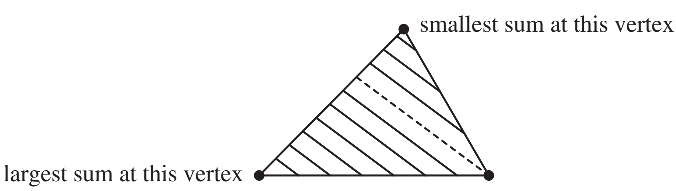
All right, ready to get stranded on a triangular island?

For the sake of completeness we mention the following result, which was derived in [**1**]. (The arguments in this article can also be tweaked easily to yield a second proof of our results.)

Let S be the sum of the distances from a point p inside a triangle that is not equilateral to the three sides of the triangle. Then (1) the triangle can be divided into parallel line segments, called isosum segments, on which S is constant, and (2) the sums associated with different isosum segments are different.

We've seen that, in isosceles triangles, points on the base all have the same distance sum. Therefore, the isosum segments in (proper) isosceles triangles are just the segments parallel to the base.

The following picture shows the isosum segments for our example of a generic triangle.



For generic triangles we don't know of any easy rule to pin down the exact direction of the isosum segments. However, what we can say is that, in a generic triangle, as in the one shown above, the isosum segment that passes through the "medium sum" vertex will intersect the edge that connects the smallest and largest sum vertices in an interior point. This is an easy consequence of (2).

We've prepared an interactive *Mathematica* CDF file that you can use to visualize everything we've been talking about in this note. Download the file from http://www.qedcat.com/isosum.cdf and open it with either *Mathematica* or the freely available *Wolfram CDF player*.

REFERENCES

1. Elias Abboud, Viviani's theorem and its extensions, *College Math. J.* **41** (2010) 203–211. http://dx.doi.org/10.4169/074683410X488683
2. Viviani in isosceles triangle: What is it? A Mathematical Droodle. An applet with accompanying explanation on the Cut-the-knot site:
http://www.cut-the-knot.org/Curriculum/Geometry/VivianiIsosceles.shtml
3. Harold Jacobs, *Geometry: Seeing, Doing, Understanding*, Freeman, New York, 2003.
4. Ken-Ichiroh Kawasaki, Proof without words: Viviani's theorem, *Math. Mag.* **78** (2005) 213.

Summary We consider a natural twist to a beautiful one-glance proof of Viviani's theorem, and its implications for general triangles.

Bisections and Reflections

JORGEN BERGLUND
California State University, Chico
Chico, CA 95929
jjberglund@csuchico.edu

RON TAYLOR
Berry College
Mount Berry, GA 30149
rtaylor@berry.edu

In this paper we examine the reflections across the angle bisectors of polygons. When we compose these reflections some natural questions arise. What are the fixed points? When do we get the identity? Our investigation begins with a triangle.

Let ABC be any triangle, and choose a point P_0 on side $\overline{AB}$. Now reflect this point across the bisector of $\angle B$, then across the bisector of $\angle C$ and so on, always reflecting across the next angle bisector as we continue moving in a counterclockwise direction. As we iterate this process, we generate a sequence of points that lie on the lines containing the sides of the triangle. Eventually something interesting happens: We return to the starting point. In fact, we always return to P_0 on the sixth reflection.

A sequence of reflected points is shown in FIGURE 1.

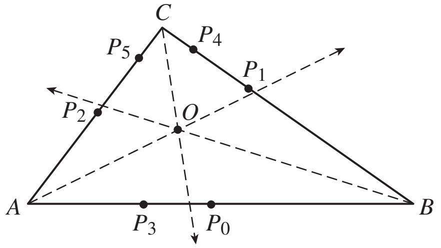


Figure 1 Reflections of a point across the angle bisectors of a triangle

We can verify that we return to P_0 after six steps by measuring the distance to each reflected point from the vertex of the angle whose bisector is the next line of reflection. (This is a directed distance; distances are negative outside the triangle beyond the selected vertex.)

Then we have the following distances as we make the first six reflections around the triangle:

$$d_0 = |P_0B|, \qquad d_1 = |P_1C| = |BC| - d_0,$$
$$d_2 = |P_2A| = |AC| - d_1, \qquad d_3 = |P_3B| = |AB| - d_2,$$
$$d_4 = |P_4C| = |BC| - d_3, \qquad d_5 = |P_5A| = |AC| - d_4,$$
$$d_6 = |P_6B| = |AB| - d_5.$$

But we can rewrite the distance d_6, for the sixth reflection, as the telescoping sum

$$d_6 = |AB| - |AC| + |BC| - |AB| + |AC| - |BC| + d_0 = d_0.$$

Math. Mag. **87** (2014) 284–290. doi:10.4169/math.mag.87.4.284. © Mathematical Association of America

Formally, we have the following result, which gives a partial answer to the question about when the composition of reflections acts like the identity.

THEOREM 1. *A point on a side of a triangle is returned to its original position by the sixth reflection across successive angle bisectors.*

The obvious question here is whether this so-called *return time* of six is a minimum. Given that we are considering a triangle, we know that the return time would need to be at least three since the point would have to make a stop on each of the other two sides before it returned. But is it possible for a point to return to its original position in exactly three reflections? Fortuitously this is possible if the initial point is one of the points of tangency between the incircle and the triangle as shown in FIGURE 2.

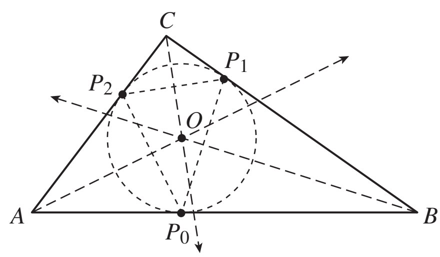


Figure 2 Reflections of tangent points of the incircle

Recall that the composition of two reflections across intersecting lines is a rotation about the point of intersection. Since the angle bisectors of $\triangle ABC$ meet at the incenter O, we see that each reflected point would lie on a circle centered at O and, at the same time, on the line containing a side of the triangle. Thus, if the point P_0 is a point of tangency between the incircle and the triangle, the reflections would also be points of tangency. Hence the point P_0 would be returned to its original position by the third reflection. Because the point returned to its starting position in a number of reflections equal to the number of sides—in this case three—we call it a *sweet spot*. This gives us the following corollary, which provides a partial answer to the question about fixed points of a composition of reflections.

COROLLARY. *A point on a triangle is a sweet spot if and only if it is a point of intersection between the incircle and the triangle.*

The authors have used the previous exploration with students, one with a class of in-service teachers and the other as a research project for pre-service teachers. The results found by the in-service teachers, including Theorem 1 and its corollary, are available in [**1**].

As an extension of this investigation, we consider the following question.

> *What happens if we consider a similar sequence of reflections in a polygon with more than three sides?*

The results here are somewhat surprising, with two different outcomes depending on the parity of the number of sides. We begin with an extension of the triangle case by considering polygons with an odd number of sides. But first we note that, while the figures provided show convex polygons, the results hold for non-convex polygons and even for self-intersecting polygons, and depend only on the parity of the number of sides. We also note that, as we compose the reflections, our convention is that we follow the vertices around the polygons in a cyclic order with a counterclockwise orientation.

Polygons with an odd number of sides

Consider a polygon with k sides. When k is odd, just as in the case of a triangle, every point on every side of the polygon is returned to its original position by the $2k$-th reflection. But unlike the triangle case, it is not necessary that every side has a sweet spot. (Recall that a sweet spot is a point whose return time is equal to the number of sides of the polygon.) However, we can determine which sides have sweet spots, and we can locate them. As before, we can calculate the distances from the points to the adjacent vertex of reflection and chase the points around the polygon as we reflect. Let $A_0A_1 \cdots A_{k-1}$ be a k-gon with side lengths $\ell_1 = |A_0A_1|, \ell_2 = |A_1A_2|, \ldots, \ell_k = |A_{k-1}A_0|$. Now we can calculate distances as before by letting $d_0 = |P_0A_1|$, and for each value of j with $1 \leq j \leq 2k$ we have

$$\begin{aligned} d_j &= |P_{j\,(\mathrm{mod}\,k)}A_{j+1\,(\mathrm{mod}\,k)}| \\ &= |A_{j\,(\mathrm{mod}\,k)}A_{j+1\,(\mathrm{mod}\,k)}| - d_{j-1} \\ &= \ell_{j+1} - d_{j-1}. \end{aligned}$$

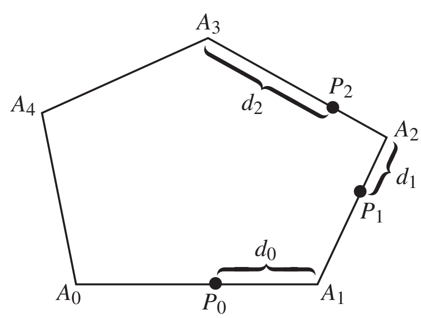


Figure 3 The first three distances

FIGURE 3 shows a pentagon with the first three distances indicated. (Recall that a negative distance indicates that the reflected point does not lie on the ray $\overrightarrow{A_mA_{m-1}}$.) As in the triangle case, if we follow the points around the polygon and write the distances as we have indicated, the final distance becomes a telescoping sum that collapses into $d_{2k} = \ell_1 - d_1 = d_0$. This means that P_{2k} lies on the same side of the polygon as P_0, and is the same distance away from A_0 as P_0. That is, $P_{2k} = P_0$. This gives us the following, more general, version of Theorem 1.

THEOREM 2. *A point P on a side of a polygon with an odd number of sides k is returned to its original point by the $2k$-th reflection.*

Now we can turn to finding potential sweet spots. In a polygon with more than three sides the angle bisectors are unlikely to be concurrent. So it is unlikely that the set of reflected points lie on a common circle. However, we can continue from the example above to see how to determine which sides do have sweet spots and where these points are located.

EXAMPLE. Consider the pentagon shown in FIGURE 4 with sides of length $\ell_1 = |A_0A_1|$, $\ell_2 = |A_1A_2|$, $\ell_3 = |A_2A_3|$, $\ell_4 = |A_3A_4|$, and $\ell_5 = |A_4A_0|$. Then if a point P_0 on side $\overline{A_0A_1}$ is to be a sweet spot, we must have $d_4 = \ell_1 - d_0$. But since we know that $d_1 = \ell_2 - d_0$, $d_2 = \ell_3 - d_1$, $d_3 = \ell_4 - d_2$, and $d_4 = \ell_5 - d_3$, we must have

$$\ell_1 - d_0 = \ell_5 - \ell_4 + \ell_3 - \ell_2 + d_0,$$

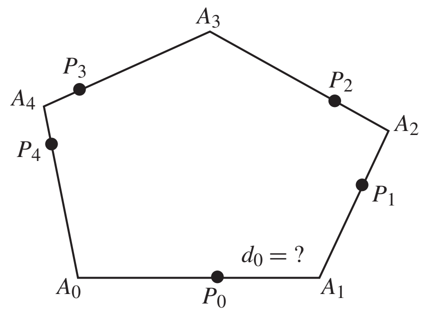


Figure 4 When is P_0 a sweet spot of a pentagon?

which can be rewritten as

$$d_0 = \frac{(\ell_1 + \ell_2 + \ell_4) - (\ell_3 + \ell_5)}{2}.$$

Hence, to determine whether a given side of the pentagon has a sweet spot, we can perform a similar calculation. Whenever this value is less than the length of the corresponding side, there is a sweet spot located at that distance from the first vertex of reflection. Our old friend the triangle inequality ensures that each side of the triangle has a sweet spot. In the general case of a polygon with an odd number of sides, we have the following result.

THEOREM 3. *Let $A_0A_1A_2\cdots A_{k-1}$ be a polygon with an odd number of sides with side lengths $\ell_1, \ell_2, \ell_3, \ldots, \ell_k$, where $|A_{j-1}A_{j\,(\mathrm{mod}\,k)}| = \ell_j$ for $1 \le j \le k$. Then side $\overline{A_0A_1}$ has a sweet spot located at a distance d from A_1, where d is given by*

$$d = \frac{(\ell_1 + \ell_2 + \ell_4 + \cdots + \ell_{k-1}) - (\ell_3 + \ell_5 + \cdots + \ell_k)}{2},$$

provided $d \in (0, \ell_1)$.

As an extension, if we consider points on the lines containing the sides of the polygon as possible periodic points, then each line has a corresponding sweet spot whose position can be calculated using the value of d from Theorem 3.

Polygons with an even number of sides

As we move to the case of a polygon with an even number of sides, we find two contrasting possibilities. The first option is that every point on every side is a sweet spot. The other outcome is that no point ever returns to its starting position. In the second case, the process of composition creates a set of points that spirals off to infinity. Serendipitously, the calculation that shows us when we have encountered the all-sweet-spot case also tells us something about how the points spiral off to infinity in the other case.

The characterization of these two cases arises from a relationship among the lengths of the sides. We define an *alternating side sum* to be an alternating sum of the lengths of the sides of a $2n$-gon,

$$S = \ell_1 - \ell_2 + \ell_3 - \cdots - \ell_{2n}.$$

The value of this sum is the device that allows us to determine which of the two cases we have encountered.

EXAMPLE. Consider the hexagon shown in FIGURE 5 with side lengths

$$\ell_1 = |A_0A_1|,\ \ell_2 = |A_1A_2|,\ \ell_3 = |A_2A_3|,\ \ell_4 = |A_3A_4|,\ \ell_5 = |A_4A_5|,\ \ell_6 = |A_5A_0|,$$

and choose a point P_0 on side $\overline{A_0A_1}$. Then, letting $d_0 = |P_0A_1|$, as we reflect the points around the hexagon we have the following sequence of distances:

$$d_1 = \ell_2 - d_0,\ d_2 = \ell_3 - d_1,\ d_3 = \ell_4 - d_2,\ d_4 = \ell_5 - d_3,\ d_5 = \ell_6 - d_4,\ d_6 = \ell_1 - d_5.$$

Because the distances are defined recursively, we can expand d_7 as

$$d_6 = \ell_1 - \ell_6 + \ell_5 - \ell_4 + \ell_3 - \ell_2 + d_0,$$

which can be rewritten as

$$d_6 - d_0 = \ell_1 - \ell_2 + \ell_3 - \ell_4 + \ell_5 - \ell_6.$$

So, $P_6 = P_0$ precisely when the alternating side sum is zero.

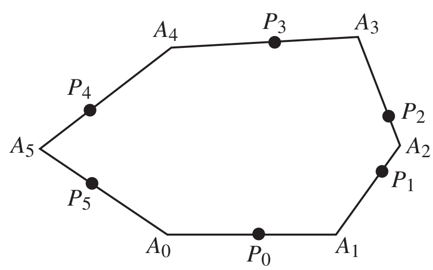


Figure 5 When is P_0 a sweet spot of a hexagon?

In the previous example the initial position of P_0 did not factor into the calculation. In fact, the only thing we knew about P_0 was that it was chosen on side $\overline{A_0A_1}$. An equivalent calculation for points on any other side yields the same result. Hence, if the alternating side sum is zero, then every point on the hexagon is a sweet spot. Similar reasoning for an arbitrary $2n$-gon gives us the following theorem.

THEOREM 4. *Let M be a polygon with an even number of sides. Then every point of M is a sweet spot if and only if an alternating side sum of M is zero.*

But what about the case when the alternating side sum is not zero? In the example with the hexagon, we notice that the signed distance from P_6 to P_0 is given by the alternating side sum—where, as is our convention, a positive value indicates that P_6 would lie on ray $\overrightarrow{P_0A_0}$, and a negative distance means that P_6 would lie on ray $\overrightarrow{P_0A_1}$. Hence, if this distance is not zero, then the collection of reflected points forms an unbounded set where every sixth point lies on the line containing one of the sides of the hexagon. Moreover, every sixth point will be the same distance away from the previous point on the same line and in the same direction away from, or closer to, the associated vertex. As always, this holds for any polygon with an even number of sides. Formally, we have the following result.

THEOREM 5. *Suppose M is a polygon with $2n$ sides and alternating side sum S. If $S \neq 0$, then no point of M is ever returned to its starting position by a sequence of reflections across successive angle bisectors. Moreover, the collection of reflected points forms an unbounded set with $|P_kP_{k+2n}| = |S|$.*

These results conform to the notion that the composition of an even number of reflections is either a rotation or a translation. The interesting thing is that by considering the lines of reflection to be the angle bisectors of a polygon, we can determine the length of the translation—thinking of a fixed point as a translation of zero length—without actually having to reflect anything.

Cyclic quadrilaterals

As noted before, in a polygon with more than three sides, there is no reason to expect that the angle bisectors are concurrent. However, in the case of a quadrilateral, there is a condition, related to our investigation, that coincides with this phenomenon. The starting point is the following interesting fact about cyclic quadrilaterals, which are quadrilaterals whose vertices lie on a common circle [**2**].

LEMMA. *A quadrilateral is cyclic if and only if the perpendicular bisectors of all its sides are concurrent.*

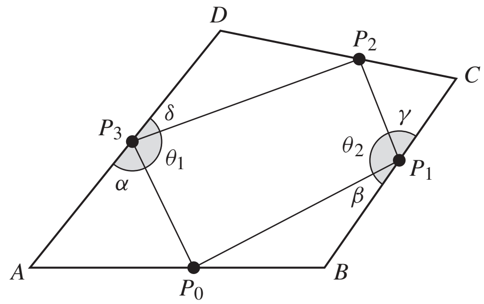


Figure 6 A quadrilateral whose angle bisectors are concurrent

EXAMPLE. Consider a quadrilateral $ABCD$ with alternating side sum zero, such as the one in FIGURE 6, and let P_0 be a point on side $\overline{AB}$. Then we know that P_0 is a sweet spot by Theorem 4. Hence, the quadrilateral $P_0P_1P_2P_3$ creates four isosceles triangles at the corners of $ABCD$ with the following angle measurements:

$$\mathrm{m}(\angle A) = \pi - 2\alpha, \quad \mathrm{m}(\angle B) = \pi - 2\beta, \quad \mathrm{m}(\angle C) = \pi - 2\gamma, \quad \mathrm{m}(\angle D) = \pi - 2\delta.$$

But then, since the sum of the interior angles of a quadrilateral is 2π, we know that

$$2\pi = (\pi - 2\alpha) + (\pi - 2\beta) + (\pi - 2\gamma) + (\pi - 2\delta),$$

which simplifies to $\pi = \alpha + \beta + \gamma + \delta$. Moreover, we can also see that the opposite angles θ_1 and θ_2 can be calculated by

$$\theta_1 = \pi - (\alpha + \delta) \quad \text{and} \quad \theta_2 = \pi - (\beta + \gamma),$$

which gives us $\theta_1 + \theta_2 = \pi$. Hence we see that the quadrilateral $P_0P_1P_2P_3$ is cyclic. Now, since the four triangles $\triangle P_0BP_1$, $\triangle P_1CP_2$, $\triangle P_2DP_3$, and $\triangle P_3AP_0$ are all isosceles, we know that the angle bisectors of $ABCD$ are the perpendicular bisectors of the sides of $P_0P_1P_2P_3$. Therefore, by the lemma, we see that the angle bisectors of $ABCD$ must be concurrent.

This gives us the following theorem.

THEOREM 6. *Let Q be a quadrilateral. Then the following are equivalent:*

(a) *The alternating side sums of Q are equal.*
(b) *The angle bisectors of Q are concurrent.*
(c) *Every point of Q is a sweet spot.*
(d) *The polygon $P_0P_1P_2P_3$ generated by the reflections is a cyclic quadrilateral.*

Since it is possible to have bisectors of adjacent angles being parallel in a polygon with more than four sides, the previous result does not generalize as the earlier ones have. However, we can see that if the angle bisectors of a $2n$-gon are concurrent, then we are back to the case in which each pair of reflections forms a rotation; and since all of the rotations are about a common point, the set of reflected points lies on a common circle, and so every point is returned to its starting point in a number of reflections equal to the number of sides. Formally, we have our final result.

THEOREM 7. *If the angle bisectors of a $2n$-gon are concurrent, then every point on the sides is a sweet spot.*

Acknowledgment The authors would like to acknowledge Professor Jim Jones of Cal State Chico for posing the original question that led to this investigation.

REFERENCES

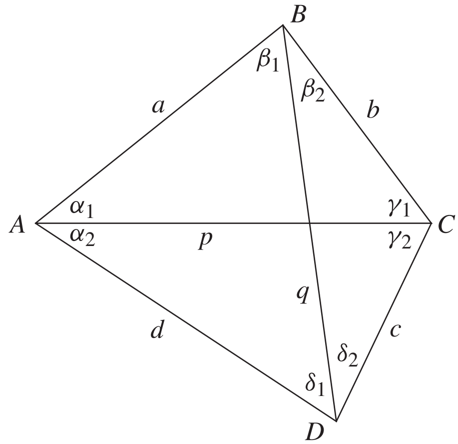
1. Jorgen Berglund, Creating a full mathematical experience in the classroom, *JOMA*, 2005. http://www.maa.org/publications/periodicals/loci/joma/creating-mathematical-experience-in-the-classroom
2. Zalman Usiskin, Jennifer Griffin, David Witonsky, and Edwin Willmore, *The Classification of Quadrilaterals: A Study in Definition*, Information Age Pub., Charlotte, NC, 2008.

Summary We consider the process of reflecting a point on the side of a polygon across successive angle bisectors. As we iterate this process, we find an interesting and sometimes periodic pattern. In particular, we show that for any polygon with an odd number of sides, a point on any side is returned to its original position by a finite number of reflections that is either one or two times the number of sides. Additionally, for a polygon with an even number of sides, a point is guaranteed to return to its original position as soon as possible or not at all.

Proof Without Words: Ptolemy's Inequality

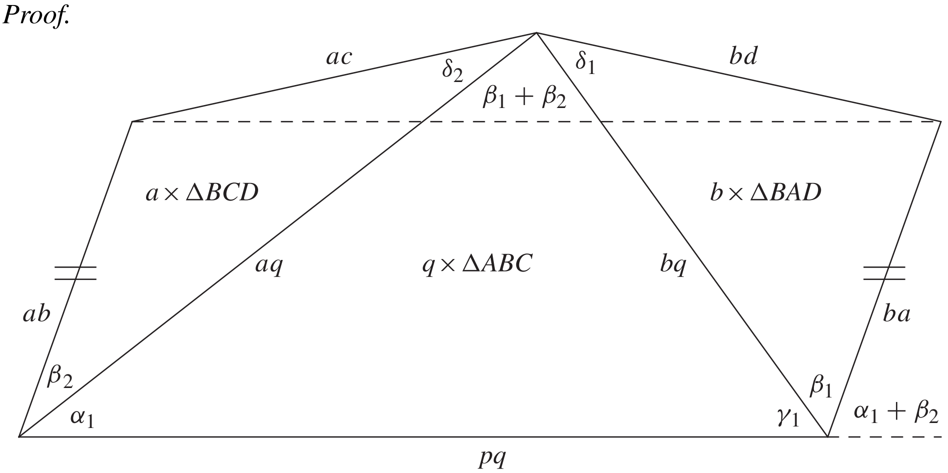
CLAUDI ALSINA
Universitat Politècnica de Catalunya

ROGER B. NELSEN
Lewis & Clark College



PTOLEMY'S INEQUALITY. In a convex quadrilateral with sides of length a, b, c, d (in that order) and diagonals of length p and q, we have $pq \le ac + bd$.

Proof.



NOTE. The angle at the top of the figure, $\delta_2 + \beta_1 + \beta_2 + \delta_1$, is drawn as being smaller than π, but the broken line representing $ac + bd$ is at least as long as the base of the parallelogram in any case. In a cyclic quadrilateral, pairs of opposite sides have sum π so that $\delta_2 + \beta_1 + \beta_2 + \delta_1 = \pi$, leading to equality:

PTOLEMY'S THEOREM. In a cyclic quadrilateral with sides of length a, b, c, d (in that order) and diagonals of length p and q, we have $pq = ac + bd$.

Summary Ptolemy: In a convex quadrilateral with sides of length a, b, c, d (in that order) and diagonals of length p and q, we have $pq \le ac + bd$.

Math. Mag. **87** (2014) 291. doi:10.4169/math.mag.87.4.291.

PROBLEMS

BERNARDO M. ÁBREGO, *Editor*
California State University, Northridge

Assistant Editors: SILVIA FERNÁNDEZ-MERCHANT, California State University, Northridge; JOSÉ A. GÓMEZ, Facultad de Ciencias, UNAM, México; EUGEN J. IONASCU, Columbus State University; ROGELIO VALDEZ, Facultad de Ciencias, UAEM, México; WILLIAM WATKINS, California State University, Northridge

PROPOSALS

To be considered for publication, solutions should be received by March 1, 2015.

1951. *Proposed by Timothy Hall, PQI Consulting, Cambridge, MA.*

Find all positive integers n for which the last three digits in base 10 of n^{100} are 376.

1952. *Proposed by Ángel Plaza, Universidad de las Palmas de Gran Canaria, Las Palmas, Spain.*

Let $\{a_n\}_{n\geq 1}$ be the sequence of real numbers defined by $a_1 = 3$ and for all $n \geq 1$, $a_{n+1} = \frac{1}{2}(a_n^2 + 1)$. Evaluate

$$\sum_{k=1}^{\infty} \frac{1}{1 + a_k}.$$

1953. *Proposed by Roberto Tauraso, Università di Roma "Tor Vergata," Roma, Italy.*

Given a graph $G = (V, E)$, a *perfect matching* M of G is a subset of the set of edges E such that every vertex $v \in V$ lies on exactly one edge in M. Prove that for each positive integer n there is a planar connected graph G whose total number of perfect matchings is equal to n.

1954. *Proposed by George Apostolopoulos, Messolonghi, Greece.*

Evaluate

$$\lim_{n\to\infty} \left(\int_1^{e^2} \left(\frac{\ln x}{x} \right)^n dx \right)^{1/n}.$$

Math. Mag. **87** (2014) 292–298. doi:10.4169/math.mag.87.4.292. © Mathematical Association of America

We invite readers to submit problems believed to be new and appealing to students and teachers of advanced undergraduate mathematics. Proposals must, in general, be accompanied by solutions and by any bibliographical information that will assist the editors and referees. A problem submitted as a Quickie should have an unexpected, succinct solution. Submitted problems should not be under consideration for publication elsewhere.

Solutions should be written in a style appropriate for this MAGAZINE.

Solutions and new proposals should be mailed to Bernardo M. Ábrego, Problems Editor, Department of Mathematics, California State University, Northridge, 18111 Nordhoff St, Northridge, CA 91330-8313, or mailed electronically (ideally as a LaTeX or pdf file) to mathmagproblems@csun.edu. All communications, written or electronic, should include **on each page** the reader's name, full address, and an e-mail address and/or FAX number.

1955. *Proposed by Elton Bojaxhiu, Kriftel, Germany and Enkel Hysnelaj, University of Technology, Sydney, Australia.*

Let ABC be a triangle. Let A_1, B_1, and C_1 be points in the interior of $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively, such that $\overline{AA_1}$, $\overline{BB_1}$, and $\overline{CC_1}$ are concurrent. Let A_2, B_2, and C_2 be the respective points of intersection of the pairs of lines (B_1C_1, BC), (C_1A_1, CA), and (A_1B_1, AB). Let P be a point in the plane. Show that if two of the angles $\angle A_1PA_2$, $\angle B_1PB_2$, and $\angle C_1PC_2$ are right angles, then the third angle must also be a right angle.

Quickies

Answers to the Quickies are on page 298.

Q1043. *Proposed by Michael Goldenberg, The Ingenuity Project, Baltimore Polytechnic Institute, Baltimore MD and Mark Kaplan, Towson University, Towson, MD.*

It looks like the values of Euler's prime-generating polynomial $P(n) = n^2 + n + 41$ are mostly prime numbers or numbers with a few prime divisors. In particular, if $1 \leq n \leq 100$, then 87% of the values of $P(n)$ are prime and 13% of the values can be factored into a product of two prime numbers. Prove that, nevertheless, there is a sequence n_k of natural numbers such that the prime factorization of $P(n_k)$ has no less than k factors.

Q1044. *Proposed by Michel Bataille, Rouen, France.*

Prove that the following inequality holds for all $a \in (0, 1)$:

$$\int_a^1 \frac{e^t}{t} dt \geq e \cdot \sinh(1 - a).$$

Solutions

Rearranging two sequences **October 2013**

1926. *Proposed by H. A. ShahAli, Tehran, Iran.*

Let n be a positive integer. For $1 \leq i \leq n$, let x_i and y_i be positive real numbers such that $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i = 1$. Prove that there exist some rearrangements $X_1, \ldots, X_n$ of $x_1, \ldots, x_n$ and $Y_1, \ldots, Y_n$ of $y_1, \ldots, y_n$ such that $\sum_{i=1}^k X_i/Y_i \geq k$ for all $1 \leq k \leq n$.

Solution by GWstat Problem Solving Group, Department of Statistics, The George Washington University, Washington, DC.

Let $X_1, \ldots, X_n$ be an arrangement in decreasing order of $x_1, \ldots, x_n$, and $Y_1, \ldots, Y_n$ be an arrangement in increasing order of $y_1, \ldots, y_n$. Noticing that $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i = 1$, we can find an integer $m \in \{1, \ldots, n\}$, such that $X_i \geq Y_i$ for all $1 \leq i \leq m$ and $X_i \leq Y_i$ for all $m < i \leq n$.

For $1 \leq k \leq m$, we have

$$\sum_{i=1}^k \frac{X_i}{Y_i} \geq \sum_{i=1}^k \frac{X_i}{X_i} = k.$$

For $m < k \leq n$, noticing that

$$\sum_{i=1}^m (X_i - Y_i) = \sum_{i=1}^m (Y_i - X_i) \geq \sum_{i=1}^k (Y_i - X_i),$$

we have

$$\sum_{i=1}^{k}\frac{X_i}{Y_i}=\sum_{i=1}^{m}\left(1+\frac{X_i-Y_i}{Y_i}\right)+\sum_{i=m+1}^{k}\left(1-\frac{Y_i-X_i}{Y_i}\right)$$
$$\geq k+\frac{1}{Y_{m+1}}\left(\sum_{i=1}^{m}(X_i-Y_i)-\sum_{i=m+1}^{k}(Y_i-X_i)\right)\geq k.$$

Thus, for any $1\leq k\leq n$, we have $\sum_{i=1}^{k} X_i/Y_i\geq k$, and the chosen rearrangement satisfies the required conditions.

Also solved by Michel Bataille (France), Elton Bojaxhiu (Albania) and Enkel Hysnelaj (Australia), Darin Brown, Paul Budney, Robert Calcaterra, Con Amore Problem Group (Denmark), Bill Cowieson, Robert L. Doucette, Eugene A. Herman, The Iowa State University Student Problem Solving Group, Hidefumi Katsuura, John C. Kieffer, Elias Lampakis (Greece), Northwestern University Math Problem Solving Group, Moubinool Omarjee (France), Pittsburg State University Problem Solving Group, Ángel Plaza and José M. Pacheco (Spain), Jeff Rosoff, Joel Schlosberg, Edward Schmeichel, Achilleas Sinefakopoulos (Greece), Traian Viteam (South Africa), Haohao Wang and Jerzy Wojdylo, and the proposer.

The prime remains in the numerator **October 2013**

1927. *Proposed by Azkar Dzhumadil'daev, Institute of Mathematics, Almaty, Kazakhstan.*

Let $p>3$ be a prime number. Prove that the numerator of the fraction

$$\sum_{n=3}^{p-1}\frac{n^3}{(n-1)(n-2)}$$

is divisible by p.

Solution by Achilleas Sinefakopoulos, M.N. Raptou Private High-School, Larissa, Greece.
The claim is easily seen to be true for $p=5$, so let $p>5$. Since

$$\frac{n^3}{(n-1)(n-2)}=n+3+\frac{8}{n-2}-\frac{1}{n-1},$$

for all $n\geq 3$, then for the fraction a/b in question we obtain

$$\frac{a}{b}=\sum_{n=3}^{p-1}(n+3)+\sum_{n=3}^{p-1}\frac{8}{n-2}-\sum_{n=3}^{p-1}\frac{1}{n-1}$$
$$=\left(\frac{p(p+5)}{2}-12\right)+\sum_{n=1}^{p-3}\frac{8}{n}-\sum_{n=1}^{p-2}\frac{1}{n}=\frac{p(p+5)}{2}+\sum_{n=3}^{p-3}\frac{8}{n}-\sum_{n=2}^{p-2}\frac{1}{n}. \quad (1)$$

Now write $p=2q+1$ and let $1\leq k\leq q$. By adding the first fraction to the last one, the second to the second-to-last one, and so on, we obtain

$$\sum_{n=k}^{p-k}\frac{1}{n}=\left(\frac{1}{k}+\frac{1}{p-k}\right)+\left(\frac{1}{k+1}+\frac{1}{p-k-1}\right)+\cdots+\left(\frac{1}{q}+\frac{1}{q+1}\right)$$
$$=\sum_{n=k}^{q}\frac{p}{n(p-n)}.$$

This combined with (1) yields

$$\frac{a}{b} = \frac{p(p+5)}{2} + \sum_{n=3}^{q} \frac{8p}{n(p-n)} - \sum_{n=2}^{q} \frac{p}{n(p-n)} = \frac{pr}{(p-2)!}$$

for some integer r. Hence $prb = ((p-2)!)a$, and so, p divides $((p-2)!)a$. As p is a prime number, $(p-2)!$ is coprime to p. Accordingly, p must divide a.

Also solved by Michel Bataille (France), Brian D. Beasley, Elton Bojaxhiu (Albania) and Enkel Hysnelaj (Australia), Darin Brown, Robert Calcaterra, Con Amore Problem Group (Denmark), Neil Curwen (United Kingdom), Robert L. Doucette, Dmitry Fleischman, Natacha Fontes-Merz, Michael Goldenberg and Mark Kaplan, GWstat Problem Solving Group, Ahmad Habil (Syria), Eugene A. Herman, The Iowa State University Student Problem Solving Group, John C. Kieffer, Omran Kouba (Syria), Elias Lampakis (Greece), Kee-Wai Lau (China), Rick Mabry, James Magliano, Peter McPolin (Northern Ireland), Donald J. Moore, Northwestern University Math Problem Solving Group, Moubinool Omarjee (France), Herman Roelants (Belgium), Joel Schlosberg, Nicholas C. Singer, Digby Smith (Canada), Traian Viteam (South Africa), Michael Vowe (Switzerland), Edward T. White, and the proposer.

An Erdős–Mordell type inequality **October 2013**

1928. *Proposed by George Apostolopoulos, Messolonghi, Greece.*

Let P be an arbitrary point inside $\triangle ABC$. Let α, β, and γ be the distances from P to the sides $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$, respectively. Prove that

$$(PA + PB + PC)\left(\frac{PA}{\beta\gamma} + \frac{PB}{\alpha\gamma} + \frac{PC}{\alpha\beta}\right) \geq 36.$$

Solution by Michel Bataille, Rouen, France.

Let $A = \angle BAC$, $B = \angle CBA$, $C = \angle ACB$, $u = \angle BAP$, and $v = \angle CAP$. From $\sin u = \gamma/PA$ and $\sin v = \beta/PA$ we deduce that $PA^2 = \beta\gamma/(\sin u \sin v)$. Since the function $x \mapsto \ln(\sin x)$ is concave downward on $(0, \pi)$ (its second derivative $x \mapsto -\csc^2 x$ is negative), Jensen's Inequality yields

$$\ln(\sin u) + \ln(\sin v) \leq 2\ln\left(\sin\frac{u+v}{2}\right) = 2\ln\left(\sin\frac{A}{2}\right),$$

and so $\sin u \sin v \leq \sin^2(A/2)$. From this inequality and by similarity, it follows that

$$PA \geq \frac{\sqrt{\beta\gamma}}{\sin\frac{A}{2}}, \quad PB \geq \frac{\sqrt{\gamma\alpha}}{\sin\frac{B}{2}}, \quad \text{and} \quad PC \geq \frac{\sqrt{\alpha\beta}}{\sin\frac{C}{2}}.$$

Letting $L = (PA + PB + PC)(PA/\beta\gamma + PB/\gamma\alpha + PC/\alpha\beta)$, we deduce that

$$L \geq \left(\frac{\sqrt{\beta\gamma}}{\sin\frac{A}{2}} + \frac{\sqrt{\gamma\alpha}}{\sin\frac{B}{2}} + \frac{\sqrt{\alpha\beta}}{\sin\frac{C}{2}}\right)\left(\frac{1}{\sqrt{\beta\gamma}\sin\frac{A}{2}} + \frac{1}{\sqrt{\gamma\alpha}\sin\frac{B}{2}} + \frac{1}{\sqrt{\alpha\beta}\sin\frac{C}{2}}\right)$$
$$\geq \left(\frac{1}{\sin\frac{A}{2}} + \frac{1}{\sin\frac{B}{2}} + \frac{1}{\sin\frac{C}{2}}\right)^2,$$

where the second inequality follows from the Cauchy-Schwarz Inequality. Finally, note that the function $x \mapsto 1/\sin x$ is concave upwards on the interval $(0, \pi/2)$. Hence, by Jensen's Inequality again,

$$\frac{1}{\sin\frac{A}{2}} + \frac{1}{\sin\frac{B}{2}} + \frac{1}{\sin\frac{C}{2}} \geq \frac{3}{\sin\left(\frac{A/2+B/2+C/2}{3}\right)} = \frac{3}{\sin(\pi/6)} = 6.$$

It follows that $L \geq 6^2 = 36$.

Also solved by Arkady Alt, Elton Bojaxhiu (Albania) and Enkel Hysnelaj (Australia), Robert L. Doucette, Dmitry Fleischman, Elana C. Greenspan, GWstat Problem Solving Group, John G. Heuver, Michael Goldenberg and Mark Kaplan, Omran Kouba (Syria), Elias Lampakis (Greece), Jerry Minkus, Peter Nüesch (Switzerland), Nikolai Rangelov, Achilleas Sinefakopoulos (Greece), Earl A. Smith, Traian Viteam (South Africa), Michael Vowe (Switzerland), John Zacharias and Adam Azzam, and the proposer.

A recursive divergent sequence **October 2013**

1929. *Proposed by Ángel Plaza, Universidad de las Palmas de Gran Canaria, Las Palmas, Spain.*

Let $a > 0$ be a real number and let (x_n) be the sequence defined by the recurrence relation $x_1 = 1$ and for $n \geq 1$,

$$x_{n+1} = x_n + an \prod_{i=1}^{n} x_i^{-1/n}.$$

(a) Prove that $\lim_{n\to\infty} x_n = \infty$.

(b) Calculate $\lim_{n\to\infty} x_n / \ln n$.

Solution by The Iowa State University Student Problem Solving Group, Iowa State University, Ames, IA.

Because $a > 0$ and $x_1 = 1 > 0$, it follows from the recursion relationship that the sequence $\{x_n\}$ is increasing. We show that $x_{n+1} \geq 2\sqrt{an}$. From this it will follow that

$$\lim_{n\to\infty} x_n \geq \lim_{n\to\infty} 2\sqrt{a(n-1)} = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{x_n}{\ln n} \geq \lim_{n\to\infty} \frac{2\sqrt{a(n-1)}}{\ln n} = \infty.$$

To prove the desired inequality, we use that $\{x_n\}$ is increasing and the Arithmetic Mean–Geometric Mean Inequality as follows:

$$\begin{aligned} x_{n+1} &= xn + an \prod_{i=1}^{n} x_i^{-1/n} \\ &\geq x_n + an \prod_{i=1}^{n} x_n^{-1/n} = x_n + anx_n^{-1} \geq 2\sqrt{x_n anx_n^{-1}} = 2\sqrt{an}. \end{aligned}$$

Editor's Note. Some of our readers wondered about the order of magnitude of the sequence $\{x_n\}$. Omran Kouba proved that $x_n = O(n)$. GWstat Problem Solving Group and Achilleas Sinefakopoulos noted that if $\lim_{n\to\infty} x_n/n$ exists, then it is equal to $\sqrt{ae}$.

Also solved by Michel Bataille (France), Robert Calcaterra, Dmitry Fleischman, GWstat Problem Solving Group, Omran Kouba (Syria), Elias Lampakis (Greece), Moubinool Omarjee (France), Paolo Perfetti (Italy), Edward Schmeichel, Achilleas Sinefakopoulos (Greece), Traian Viteam (South Africa), and Haohao Wang and Jerzy Wojdylo.

A zeta series **October 2013**

1930. *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.*

Find the value of

$$\sum_{n=2}^{\infty} (-1)^n n(n - \zeta(2) - \zeta(3) - \cdots - \zeta(n)),$$

where ζ denotes the Riemann zeta function defined by $\zeta(z) = \sum_{n=1}^{\infty} 1/n^z$ for $z \in \mathbb{C}$ with $\Re(z) > 1$.

Solution by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria.

The answer is $5/4 - \pi^2/12$. First note that

$$\begin{aligned}\sum_{k=2}^{n}\zeta(k) &= \sum_{k=2}^{n}\sum_{j=1}^{\infty}\frac{1}{j^k} = \sum_{j=1}^{\infty}\sum_{k=2}^{n}\frac{1}{j^k}\\ &= n-1+\sum_{j=2}^{\infty}\frac{1/j^2-1/j^{n+1}}{1-1/j} = n-1+\sum_{j=2}^{\infty}\left(\frac{1}{j(j-1)}-\frac{1}{j^n(j-1)}\right)\\ &= n-\sum_{j=2}^{\infty}\frac{1}{j^n(j-1)},\end{aligned}$$

where we used the fact that $\sum_{j=2}^{\infty} 1/(j(j-1)) = 1$. It follows that

$$A_n := n-\zeta(2)-\zeta(3)-\cdots-\zeta(n) = \sum_{j=2}^{\infty}\frac{1}{j^n(j-1)}.$$

Using the fact that $x/(1-x)^2 = \sum_{n=1}^{\infty} nx^n$ for $|x|<1$ we get

$$\begin{aligned}\sum_{n=2}^{\infty} nA_n &= \sum_{n=2}^{\infty}\sum_{j=2}^{\infty}\frac{n}{j^n(j-1)} = \sum_{j=2}^{\infty}\sum_{n=2}^{\infty}\frac{n}{j^n(j-1)}\\ &= \sum_{j=2}^{\infty}\frac{1}{j-1}\left(\frac{1/j}{(1-1/j)^2}-\frac{1}{j}\right) = \sum_{j=1}^{\infty}\left(\frac{1}{j^2}+\frac{1}{j^3}-\frac{1}{j(j+1)}\right)\\ &= \zeta(3)+\zeta(2)-1.\end{aligned}$$

Thus the series $\sum(-1)^n nA_n$ is absolutely convergent, and thus exchanging the order of summation is justified in what follows:

$$\begin{aligned}\sum_{n=2}^{\infty}(-1)^n nA_n &= \sum_{n=2}^{\infty}\sum_{j=2}^{\infty}\frac{(-1)^n n}{j^n(j-1)} = \sum_{j=2}^{\infty}\sum_{n=2}^{\infty}\frac{(-1)^n n}{j^n(j-1)}\\ &= \sum_{j=2}^{\infty}\frac{1}{j-1}\left(\frac{-1/j}{(1+1/j)^2}+\frac{1}{j}\right) = \sum_{j=2}^{\infty}\left(\frac{1}{j(j-1)}-\frac{j}{(j-1)(j+1)^2}\right)\\ &= 1-\sum_{j=2}^{\infty}\frac{j}{(j-1)(j+1)^2}.\end{aligned}$$

But

$$\frac{4j}{(j-1)(j+1)^2} = \frac{1}{j-1}-\frac{1}{j+1}+\frac{2}{(j+1)^2},$$

hence

$$\begin{aligned}\sum_{n=2}^{\infty}(-1)^n nA_n &= 1-\frac{1}{4}\sum_{j=2}^{\infty}\left[\left(\frac{1}{j-1}-\frac{1}{j+1}\right)+\frac{2}{(j+1)^2}\right]\\ &= 1-\frac{1}{4}\left(\frac{3}{2}+2\left(\zeta(2)-\frac{5}{4}\right)\right) = \frac{5}{4}-\frac{\pi^2}{12}.\end{aligned}$$

Also solved by Michel Bataille (France), Gerald E. Bilodeau, Khristo Boyadzhiev, Bruce S. Burdick, Robert Calcaterra, Minh Can, Hongwei Chen, Con Amore Problem Group (Denmark), Robert L. Doucette, Dmitry Fleischman, GWstat Problem Solving Group, Eugene A. Herman, The Iowa State University Student Problem Solving Group, Michael Goldenberg and Mark Kaplan, Elias Lampakis (Greece), Paolo Perfetti (Italy), Ángel Plaza (Spain), Rob Pratt, Henry Ricardo, Edward Schmeichel, Achilleas Sinefakopoulos (Greece), Nicholas C. Singer, Zeda Thomas, and Michael Vowe (Switzerland).

Answers

Solutions to the Quickies from page 293.

A1043. Let $n_1 = 1$ and $n_{k+1} = n_k^2 + 40$ for $k \geq 1$. Note that

$$\begin{aligned} P(n_{k+1}) &= P(n_k^2 + 40) = n_k^4 + 82n_k^2 + 41^2 - n_k^2 \\ &= (n_k^2 + 41)^2 - n_k^2 = (n_k^2 - n_k + 41)(n_k^2 + n_k + 41) = P(-n_k)P(n_k). \end{aligned}$$

Continuing the recursion gives

$$P(n_{k+1}) = P(-n_k)P(-n_{k-1})P(-n_{k-2}) \cdots P(-n_2)P(-n_1)P(n_1),$$

and the conclusion follows since each factor in the right-hand side is greater than one.

A1044. The functions $x \mapsto e^x/x$ and $x \mapsto xe^x$ are respectively decreasing and increasing on $(0, 1]$, hence $I \leq 0$ where

$$I = \int_a^1 \int_a^1 \left(\frac{e^x}{x} - \frac{e^y}{y}\right)(xe^x - ye^y)\,dx\,dy.$$

The result follows because

$$\begin{aligned} I &= \int_a^1 \int_a^1 \left(e^{2x} - ye^y\frac{e^x}{x} - xe^x\frac{e^y}{y} + e^{2y}\right)dx\,dy \\ &= 2(1-a)\int_a^1 e^{2x}dx - 2\int_a^1 \frac{e^x}{x}dx \int_a^1 ye^y dy \\ &= (1-a)(e^2 - e^{2a}) - 2(1-a)e^a \int_a^1 \frac{e^t}{t}dt \\ &= 2(1-a)e^a\left(e \cdot \frac{1}{2}(e^{1-a} - e^{a-1}) - \int_a^1 \frac{e^t}{t}dt\right) \\ &= 2(1-a)e^a\left(e \cdot \sinh(1-a) - \int_a^1 \frac{e^t}{t}dt\right). \end{aligned}$$

The inequality is sharp because by L'Hospital's rule

$$\lim_{a \to 1^-} \frac{\int_a^1 e^t/t\,dt}{\sinh(1-a)} = \lim_{a \to 1^-} \frac{e^a/a}{\cosh(1-a)} = e.$$

REVIEWS

PAUL J. CAMPBELL, *Editor*
Beloit College

Assistant Editor: Eric S. Rosenthal, West Orange, NJ. Articles, books, and other materials are selected for this section to call attention to interesting mathematical exposition that occurs outside the mainstream of mathematics literature. Readers are invited to suggest items for review to the editors.

Alexander, Amir, *Infinitesimal: How a Dangerous Mathematical Theory Shaped the Modern World*, Scientific American / Farrar, Straus and Giroux, 2014. 352 pp, $27. ISBN 978-0-374-1768-5. Extract: The secret spiritual history of calculus, *Scientific American* 310 (4) (April 2014) 82–85, http://www.scientificamerican.com/article/the-secret-spiritual-history-of-calculus/, with subsequent letters at 311 (2) (August 2014) 6–7.

I have urged elsewhere that calculus be "intellectualized" and re-embedded in the historical context of culture and science, thereby returning it to the fold of the liberal arts. Central to that rebirth could be the question of infinitesimal numbers: Do they exist? If so, how should we calculate with them? Even if not, are they useful pedagogical constructs? Author Alexander situates "indivisibles" at the heart of a battle of the Jesuits against the "disorder" of established truth that infinitesimals represented. However, the emphasis in the book is definitely on religious conflicts, and one must wonder if infinitesimals were really so important or were just a pawn and sideshow in those struggles. Mathematics occupies only about 30 pages of the book, which ends its tale around 1650—before the era of Newton, Leibniz, and George Berkeley's objections, and long before the legitimation in the 20th century of number systems involving infinitesimals.

Avigad, Jeremy, and John Harrison, Formally verified mathematics, *Communications of the Association for Computing Machinery* 57 (4) (April 2014) 66–75. http://cacm.acm.org/magazines/2014/4/173219-formally-verified-mathematics/fulltext.

The Flyspeck Project: Announcement of Completion (16 August 2014), http://code.google.com/p/flyspeck/wiki/AnnouncingCompletion.

Aron, Jacob, Proof confirmed of 400-year-old fruit-stacking problem, *New Scientist* (12 August 2014). http://www.newscientist.com/article/dn26041-proof-confirmed-of-400yearold-fruitstacking-problem.html.

Wolfram, Stephen, Computational knowledge and the future of pure mathematics, http://blog.stephenwolfram.com/2014/08/computational-knowledge-and-the-future-of-pure-mathematics.

Are you necessary? "[I]t is now possible to achieve complete formalization [of mathematical proofs] in practice," claim Avigad and Harrison. The uncertainty about Thomas Hales's 1998 proof of the Kepler conjecture (about the densest packing of balls in euclidean 3-space)—a situation that may occur more frequently in the future—was resolved by Hales's "Flyspeck" project to formalize his proof. Says Hales, "This technology cuts the mathematical referees out.... Their opinion ... no longer matters." Too late for the 12 who spent four years reviewing Hales's earlier proof but by the end could be only 99% certain that that proof was correct! Wolfram plans to "create a precise symbolic language that captures the concepts and constructs of pure mathematics" that will become part of a new "workflow" toward "mathematical wisdom." He describes a pilot project that synthesized all known research about continued fractions and actually generated results that filled some computational gaps. What would it take to "curate the complete literature of mathematics"? Mathematicians, of course, to guide the process.

Math. Mag. **87** (2014) 299–300. doi:10.4169/math.mag.87.4.299.

Ellenberg, Jordan, *How Not to Be Wrong: The Power of Mathematical Thinking*, Penguin Press, 2014; 468 pp, $27.95. ISBN 978-1-59420-522-4.

Can this book live up to its dust cover: "unveils the hidden beauty and logic of the world and puts math's power in our hands." *It does.* It is by far the best popular book for giving a sense of what mathematics is about, and both why and how it is important. Following his theme of "mathematics is the extension of common sense by other means," Ellenberg writes informally, engagingly, sensibly, and profoundly, about all kinds of mathematics (but mainly about statistics and probability). I particularly enjoyed his interpretation of correlation in terms of the angle between two vectors, the use of finite geometries in buying lottery tickets, and his proof of the Buffon noodle [sic] theorem. Occasionally I would go further: He writes, "Dividing one number by another is mere computation; figuring out *what* you should divide by *what* is mathematics"; I would say that discovering that you should *divide at all* is where the mathematics lies. [There are clues that the book was composed in LaTeX yet someone must have decided that the reading public could not stand seeing the mathematical quantities appear in math italic. Too bad.]

Astounding: $1 + 2 + 3 + 4 + 5 + \cdots = -1/12$, https://www.youtube.com/watch?v=w-I6XTVZXww.

Padilla, Tony, What do we get if we sum all the natural numbers?, https://www.nottingham.ac.uk/~ppzap4/response.html.

Tao, Terence, The Euler-Maclaurin formula, Bernoulli numbers, the zeta function, and real-variable analytic continuation, http://terrytao.wordpress.com/2010/04/10/the-euler-maclaurin-formula-bernoulli-numbers-the-zeta-function-and-real-variable-analytic-continuation/.

Well, as if $.9999\cdots = 1$ doesn't regularly cause enough chronic commotion on math discussion boards, here is a new trope: $\sum_{n=1}^{\infty} n = -1/12$. The YouTube video (by a physicist) claims, "this result is used in many areas of physics." It exhibits p. 22 of Joseph Polchinski's *String Theory*, vol. 1, which writes not $\sum_{n=1}^{\infty} n = -1/12$ but the slightly less disagreeable $\sum_{n=1}^{\infty} n \to -1/12$, calls it an "odd result" of preserving Lorentz invariance in renormalization, and just proceeds onward (so maybe "used in physics" is a stretch). Padilla remarks that the result can be arrived at by "perfectly legitimate manipulations" in connection with divergent series and analytic continuation. However, he also remarks, "having read the comments, tweets and blogs, if I were to ask again 'what do we get if we sum the natural numbers?' I think my answer would now be 'we upset people.'" And not just mathematicians—there have been 2.5 million views of the video so far. If you're skeptical, you don't want to ask what $\sum_{n=1}^{\infty} n^2$ equals; but first check out Tao's "elementary" interpretation.

Kucharski, Adam, Math's beautiful monsters, *Nautilus* No. 3 (Spring 2014) 18–25. Extract at http://nautil.us/issue/11/light/maths-beautiful-monsters.

Nautilus is an exciting beautiful new quarterly literary magazine of science, with perspectives from culture and philosophy. Each printed quarterly comprises three monthly issues, each devoted to a single theme. The article by Kucharski, part of a general topic of "Monsters," attracted my attention for its suspenseful history of the discovery of continuous nowhere-differentiable functions. That article is in a column "Numbers" that often features mathematics; the column has offered a reconstructed interview with Benoit Mandelbrot, modeling of traffic jams, transportation optimization, Kolmogorov's development of probability, Yitang "Tom" Zhang's progress on the twin prime conjecture, and the value of cheating at team tennis tournaments.

Su, Francis Edward, The lesson of grace in teaching, http://mathyawp.blogspot.com/2013/01/the-lesson-of-grace-in-teaching.html. Reprinted in *The Best Writing on Mathematics 2014*, edited by Mircea Pitici, 188–197. Princeton University Press, 2014; $24.95(P). ISBN 978-0-691-16417-5.

Francis Su, president-elect of the Mathematical Association of America, writes about sense of self ("Your accomplishments are not what make you a worthy human being") and the need to show others—your students, in particular—grace ("Good things you didn't earn or deserve, but you're getting them anyway"). This essay should become a classic for all teachers!

NEWS AND LETTERS

43rd USA Mathematical Olympiad
5th USA Junior Mathematical Olympiad

JACEK FABRYKOWSKI
Youngstown State University
Youngstown, OH 44555
jfabrykowski@ysu.edu

STEVEN R. DUNBAR
MAA American Mathematics Competitions
University of Nebraska-Lincoln
Lincoln, NE 68588-0658
sdunbar@maa.org

This year the Committee on the American Mathematics Competitions offered the USA Junior Mathematical Olympiad (USAJMO) for the fifth time for students in 10th grade and below. Our experience of the last four years shows that it provides a nicely balanced link between the computational character of the AIME problems and the proof oriented problems of the USAMO. This year the competitions took place on April 29 and 30. The USA Junior Mathematical Olympiad contained three problems for each of two days, with an allowed time of 4.5 hours each day—the same as the USAMO. The two exams had two common problems: In the first day problem JMO3 was the same as USAMO2 and in the second day JMO5 and USAMO4 were identical.

USAMO Problems

1. Let a, b, c, d be real numbers such that $b - d \geq 5$ and all zeros x_1, x_2, x_3, and x_4 of the polynomial $P(x) = x^4 + ax^3 + bx^2 + cx + d$ are real. Find the smallest value the product $(x_1^2 + 1)(x_2^2 + 1)(x_3^2 + 1)(x_4^2 + 1)$ can take.
2. Let $\mathbb{Z}$ be the set of integers. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that
$$xf(2f(y) - x) + y^2 f(2x - f(y)) = \frac{f(x)^2}{x} + f(yf(y))$$
for all $x, y \in \mathbb{Z}$ with $x \neq 0$.
3. Prove that there exists an infinite set of points
$$\ldots,\ P_{-3},\ P_{-2},\ P_{-1},\ P_0,\ P_1,\ P_2,\ P_3,\ \ldots$$
in the plane with the following property: For any three distinct integers a, b and c, points P_a, P_b and P_c are collinear if and only if $a + b + c = 2014$.
4. Let k be a positive integer. Two players A and B play a game on an infinite grid of regular hexagons. Initially all the grid cells are empty. Then the players alternately take turns with A moving first. In his move, A may choose two adjacent hexagons in the grid which are empty and place a counter in both of them. In his move, B may choose any counter on the board and remove it. If at any time there are k consecutive grid cells in a line all of which contain a counter, A wins. Find the minimum value of k for which A cannot win in a finite number of moves, or prove that no such minimum value exists.

Math. Mag. **87** (2014) 301–309. doi:10.4169/math.mag.87.4.301.

5. Let ABC be a triangle with orthocenter H and let P be the second intersection of the circumcircle of triangle AHC with the internal bisector of the angle $\angle BAC$. Let X be the circumcenter of triangle APB and Y the orthocenter of triangle APC. Prove that the length of segment XY is equal to the circumradius of triangle ABC.
6. Prove that there is a constant $c > 0$ with the following property: If a, b, n are positive integers such that $\gcd(a+i, b+j) > 1$ for all $i, j \in \{0, 1, \ldots, n\}$, then

$$\min\{a, b\} > c^n \cdot n^{n/2}.$$

USAMO Solutions

1. The smallest value is 16.

Let $X = (x_1^2+1)(x_2^2+1)(x_3^2+1)(x_4^2+1)$. As a symmetrical polynomial in the roots of $P(x)$, X must be expressible as a polynomial in the coefficients. A long algebraic argument starting with Vieta's relations gives the correct expression:

$$X = (x_1^2+1)(x_2^2+1)(x_3^2+1)(x_4^2+1) = (b-d-1)^2 + (c-a)^2.$$

When $b - d \geq 5$, the last expression must be at least $4^2 + 0^2 = 16$.

Alternatively, we may notice that

$$\begin{aligned}(x_1^2+1)(x_2^2+1)(x_3^2+1)(x_4^2+1) &= P(i)P(-i)\\ &= ((1-b+d) + i(c-a))(1-b+d-i(c-a))\\ &= (b-d-1)^2 + (c-a)^2 \geq 16,\end{aligned}$$

with equality when $b - d = 5$ and $a = c$. The value $X = 16$ is realized when $x_1 = x_2 = x_3 = x_4 = 1$ and $P(x) = x^4 - 4x^3 + 6x^2 - 4x + 1$.

This problem and the solutions were suggested by Titu Andreescu.

2. Let f be a function satisfying the given identity. Let p be a prime. Since p divides $f(p)^2$, p divides $f(p)$ and so p divides $f(p)^2/p$. Taking $y = 0$ and $x = p$, we deduce that p divides $f(0)$. As p is arbitrary, we must have $f(0) = 0$. Next, take $y = 0$ to obtain $xf(-x) = f(x)^2/x$. Replacing x by $-x$, and combining the two relations reveals that for every x, either $f(x) = f(-x) = 0$ or $f(x) = f(-x) = x^2$.

In fact, the functions $f(x) = 0$ (for all x) and $f(x) = x^2$ (for all x) satisfy the identity. Can the cases be mixed?

Suppose that there exists $y \neq 0$ such that $f(y) = 0$. With this value of y, the identity becomes $xf(-x) + y^2 f(2x) = f(x)^2/x$, yielding $y^2 f(2x) = 0$ for all x and so f vanishes on even integers.

If $y \equiv 3 \pmod 4$ and $y \neq -1$, we can choose $x = (y^2+y)/2$. Then x is a non-zero even integer and for these values of x and y, the identity simplifies to

$$y^2(y^2 + y - f(y)) = f(yf(y)).$$

If $f(y) = y^2$ this becomes $y^3 = f(y^3)$, which is impossible. So in these cases $f(y) = 0$.

If $y \equiv 1 \pmod 4$ and $y \neq 1$, we can choose $x = (y^2 - y)/2$. Then x is a non-zero even integer and for these values of x and y, and again, if $f(y) = y^2$ the identity becomes $y^3 = f(y^3)$, again impossible. So in these cases $f(y) = 0$.

Finally, if $f(1) = f(-1) = 1$ then the identity fails for $x = 2$, $y = 1$, so we must have $f(1) = f(-1) = 0$ and f vanishes for all integers. Therefore $f(x) = 0$ (for all x) and $f(x) = x^2$ (for all x) are the only functions satisfying the identity.

This problem and the solution were suggested by Titu Andreescu and Gabriel Dospinescu.

3. We claim that defining P_n to be the point with coordinates $(n, n^3 - 2014n^2)$ will satisfy the conditions of the problem. Recall that points (x_1, y_1), (x_2, y_2) and (x_3, y_3) are collinear if and only if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$$

Therefore we examine the determinant

$$\begin{vmatrix} a & a^3 - 2014a^2 & 1 \\ b & b^3 - 2014b^2 & 1 \\ c & c^3 - 2014c^2 & 1 \end{vmatrix} = \begin{vmatrix} a & a^3 & 1 \\ b & b^3 & 1 \\ c & c^3 & 1 \end{vmatrix} - 2014 \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}.$$

The first determinant on the right is a homogeneous polynomial of degree four that is divisible by $(a - b)(b - c)(c - a)$. The remaining factor has degree one, is symmetric, and yields an ab^3 term when the product is expanded, so that factor must be $(a + b + c)$. The second determinant is a homogeneous polynomial of degree three that is divisible by $(a - b)(b - c)(c - a)$, and comparing coefficients of the ab^2 term we see that this is the desired polynomial. Thus

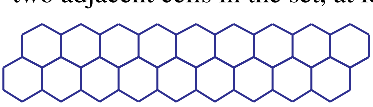
$$\begin{vmatrix} a & a^3 - 2014a^2 & 1 \\ b & b^3 - 2014b^2 & 1 \\ c & c^3 - 2014c^2 & 1 \end{vmatrix} = (a - b)(b - c)(c - a)(a + b + c - 2014).$$

It follows that for distinct a, b and c this expression equals zero if and only if $a + b + c = 2014$, as desired.

This problem was suggested by Sam Vandervelde and the solution was suggested by Razvan Gelca.

4. The answer is $k = 6$. First we show that A cannot win for $k \geq 6$. Color the grid in three colors so that no two adjacent spaces have the same color, and arbitrarily pick one color $\mathcal{C}$. Now B can play by always removing any counter that is on a space colored $\mathcal{C}$. If there is no such counter, B plays arbitrarily. Because A cannot cover two spaces colored $\mathcal{C}$ simultaneously, it is possible for B to play in this fashion. Now note that any line of six consecutive squares contains two spaces colored $\mathcal{C}$. For A to win he must cover both, but B's strategy ensures at most one space colored $\mathcal{C}$ has a counter at any time.

Now we show that A can obtain 5 counters in a row. Take a set of cells in the grid forming the shape shown below. We let A play counters only in this set of grid cells for as long as possible. Since B removes only one counter for every two that A places, the number of counters in this set will increase until A can no longer play in this set. Then, of any two adjacent cells in the set, at least one has a counter.



Consider only the top row of cells in the set, and take the lengths of each consecutive run of cells. If there are two adjacent runs that have a combined length of at least 4, then A gets 5 counters in a row by filling the space in between. Otherwise, a bit of case analysis shows that there exists a run of 1 counter which is neither the first nor last run. This single counter has an empty space on either side of it on the first row. The four spaces of the second row touching these two empty spaces all must have counters. Now A can play in the 5th cell on either side of these 4 to get 5 counters in a row. So in all cases A can win when $k \leq 5$.

This problem and the solution were suggested by Palmer Mebane.

5. It is well known that the reflection H' of the orthocenter H in the line AC lies on the circumcircle of triangle ABC. Hence, the circumcenter of triangle CAH' coincides with the circumcenter O of triangle ABC. But since H' is the reflection of H in the line AC, the triangles ACH and CAH' are symmetric with respect to AC, and the circumcenter O' of triangle ACH must be the reflection of the circumcenter of triangle CAH' in the line AC; that is, the reflection of O in the line AC.

   Now since the quadrilateral $AHPC$ is cyclic and since H and Y are the orthocenters of triangles ABC and APC, respectively, we have that

$$\angle ABC = 180^\circ - \angle AHC = 180^\circ - \angle APC = \angle AYC.$$

   Hence the point Y lies on the circumcircle of triangle ABC, and therefore $OC = OY = R$, where R denotes the circumradius of triangle ABC.

   On the other hand, noting that the lines OX, XO', and $O'O$ are the perpendicular bisectors of the segments AB, AP, and AC, respectively, we get

$$\angle OXO' = \angle BAP = \angle PAC = \angle XO'O.$$

   Thus $OO' = OX$. Combining this with $OC = OY$ and with the parallelism of the lines XO' and YC (note that these two lines are both perpendicular to AP), we conclude that the trapezoid $XYCO'$ is isosceles, and therefore $XY = O'C = OC = R$. This completes our proof.

   Note. If ABC is right-angled at A, then the statement is trivially true if we recall that the circumcenter of AB is the midpoint of AB and that the orthocenter of AC is the midpoint of AC. Then, we have that $XY = (1/2)BC = R$.

   This problem and the solution were suggested by Titu Andreescu and Cosmin Pohoata.

6. Let a, b, n be positive integers as in the statement of the problem. Let P_n be the set of prime numbers not exceeding n. We will need a lemma.

   LEMMA. *There is a positive integer n_0 such that for all $n \geq n_0$ we have*

$$\sum_{p\in P_n}\left(\frac{n}{p}+1\right)^2 < \frac{2}{3}n^2.$$

   Proof. Expanding and dividing by n^2, and observing that $|P_n| \leq n$, it suffices to prove the inequality

$$\sum_{p\in P_n}\frac{1}{p^2}+\frac{2}{n}\sum_{p\in P_n}\frac{1}{p}+\frac{1}{n} < \frac{2}{3}.$$

   Since

$$\frac{2}{n}\sum_{p\in P_n}\frac{1}{p} < \frac{2}{n}\sum_{i=2}^{n}\frac{1}{i} < \frac{2}{n}\log n,$$

   it suffices to prove the existence of a constant $r < 2/3$ such that $\sum_{p\in P_n}(1/p^2) < r$. But

$$\sum_{p\in P_n}\frac{1}{p^2} < \frac{1}{4}+\frac{1}{9}+\int_4^\infty \frac{1}{x^2}dx = \frac{1}{4}+\frac{1}{9}+\frac{1}{4} < \frac{2}{3}$$

   and we can take $r = \frac{1}{4}+\frac{1}{9}+\frac{1}{4}$. ∎

From now on we fix such n_0, and we prove the statement assuming $n \geq n_0$. For any $p \in P_n$ there are at most $(n/p) + 1$ numbers $i \in \{0, 1, \ldots, n-1\}$ such that $p \mid a + i$, and likewise for $j \in \{0, 1, \ldots, n-1\}$ such that $p \mid b + j$. Thus there are at most $((n/p) + 1)^2$ pairs (i, j) such that $p \mid \gcd(a + i, b + j)$. Using the previous lemma, we deduce that there are fewer than $(2/3)n^2$ pairs (i, j) with $i, j \in \{0, 1, \ldots, n-1\}$ such that $p \mid \gcd(a + i, b + j)$ for some $p \in P_n$. (We will not need the instances with $i = n$ or $j = n$.)

Let $N = \lceil n^2/3 \rceil$. By the above, there are at least N pairs (i, j) with $i, j \in \{0, 1, \ldots, n-1\}$ such that $\gcd(a + i, b + j)$ is not divisible by any prime in P_n. Call these pairs (i_s, j_s) for $s = 1, 2, \ldots, N$. Since $\gcd(a + i_s, b + j_s) > 1$ for each of these pairs, we may choose a prime p_s that divides both $a + i$ and $b + j$; and each $p_s > n$. The map $s \mapsto p_s$ is injective, for if $p_s = p_{s'}$, then $p_s \mid i_s - i_{s'}$, implying $i_s = i_{s'}$, and similarly $j_s = j_{s'}$, hence $s = s'$.

We conclude that $\prod_{i=0}^{n-1}(a + i)$ is a multiple of $\prod_{s=1}^{N} p_s$. Since the p_s are distinct prime numbers greater than n, it follows that,

$$(a + n)^n > \prod_{i=0}^{n-1}(a + i) \geq \prod_{s=1}^{N} p_s \geq \prod_{i=1}^{N}(n + 2i - 1).$$

Let X be this last product. Then

$$X^2 = \prod_{i=1}^{N}[(n + 2i - 1)(n + 2(N + 1 - i) - 1)] > \prod_{i=1}^{N}(2Nn) = (2Nn)^N,$$

where the inequality holds because

$$(n + 2i - 1)(n + 2(N + 1 - i) - 1) > n(2(N + 1 - i) - 1) + (2i - 1)n = 2Nn.$$

Finally

$$(a + n)^n > (2Nn)^{N/2} \geq \left(\frac{2n^3}{3}\right)^{n^2/6}.$$

Thus,

$$a \geq (2/3)^{n/6} \cdot n^{n/2} - n,$$

which is larger than $c^n \cdot n^{n/2}$ when n is large enough, for any constant $c < (2/3)^{1/6}$. The same inequality holds for b.

This shows that $\min\{a, b\} \geq c^n \cdot n^{n/2}$ as long as n is large enough. By shrinking c sufficiently, we can ensure the inequality holds for all n.

One can see that the argument is not sharp, so that the factor $n^{n/2}$ can be improved to n^{rn} for some constant r slightly larger than $1/2$. Consequently, for *any* $c > 0$, the inequality in the problem holds if n is large enough.

This problem and the solution were suggested by Titu Andreescu and Gabriel Dospinescu.

USAJMO Problems

1. Let a, b, c be real numbers greater than or equal to 1. Prove that

$$\min\left(\frac{10a^2 - 5a + 1}{b^2 - 5b + 10}, \frac{10b^2 - 5b + 1}{c^2 - 5c + 10}, \frac{10c^2 - 5c + 1}{a^2 - 5a + 10}\right) \leq abc.$$

2. Let ΔABC be a non-equilateral, acute triangle with $\angle A = 60^\circ$, and let O and H denote the circumcenter and orthocenter of ΔABC, respectively.
   (a) Prove that line OH intersects both segments AB and AC.
   (b) Line OH intersects segments AB and AC at P and Q, respectively. Denote by s and t the respective areas of triangle APQ and quadrilateral $BPQC$. Determine the range of possible values for s/t.
3. Same as USAMO 2.
4. Let $b \geq 2$ be an integer, and let $s_b(n)$ denote the sum of the digits of n when it is written in base b. Show that there are infinitely many positive integers that cannot be represented in the form $n + s_b(n)$, where n is a positive integer.
5. Same as USAMO 4.
6. Let ABC be a triangle with incenter I, incircle γ and circumcircle Γ. Let M, N, P be the midpoints of sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ and let E, F be the tangency points of γ with $\overline{CA}$ and $\overline{AB}$, respectively. Let U, V be the intersections of line EF with line MN and line MP, respectively, and let X be the midpoint of arc BAC of Γ.
   (a) Prove that I lies on ray CV.
   (b) Prove that line XI bisects $\overline{UV}$.

USAJMO Solutions

1. We start by observing that the numerators and denominators of the fractions in the statement of the problem are positive. Next, since $a \geq 1$, we have

$$a^3(a^2 - 5a + 10) - (10a^2 - 5a + 1) = (a - 1)^5 \geq 0,$$

so that

$$\frac{10a^2 - 5a + 1}{a^2 - 5a + 10} \leq a^3.$$

Similar inequalities hold for b and c, and multiplying the three inequalities gives

$$\frac{10a^2 - 5a + 1}{a^2 - 5a + 10} \cdot \frac{10b^2 - 5b + 1}{b^2 - 5b + 10} \cdot \frac{10c^2 - 5c + 1}{c^2 - 5c + 10} \leq (abc)^3.$$

For this to be true, at least one of the fractions must be less than or equal to abc, as required.

This problem and the solution were suggested by Titu Andreescu.

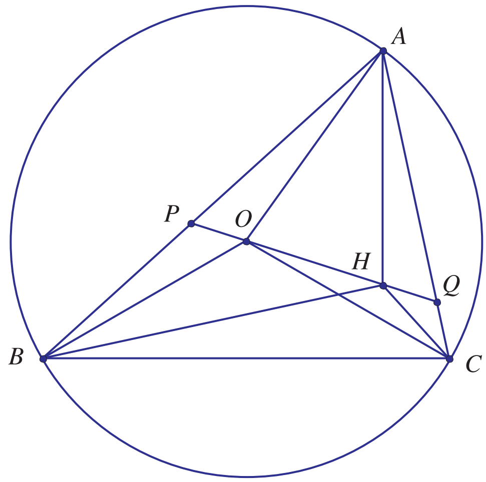
2. **(a)** Without loss of generality, we assume that $AB > AC$. Set $\beta = \angle ABC$ and $\gamma = \angle ACB$. We have $\beta < 60^\circ < \gamma$ and $\beta + \gamma = 120^\circ$.

Note that $\angle BAO = 90^\circ - \angle ACB = 90^\circ - \gamma < 90^\circ - \beta = 90^\circ - \angle ABC = \angle BAH$, and so AO lies inside $\angle BAH$. Similarly, $\angle ABO = 90^\circ - \gamma < 30^\circ = \angle ABH$, and so BO lies inside $\angle ABH$. Hence O lies inside $\triangle ABH$, and line OH intersects side AB. In the same way, $\angle CAH = 90^\circ - \gamma < 90^\circ - \beta = \angle CAO$ and $\angle ACH = 30^\circ < 90^\circ - \beta = \angle ACO$; hence H lies inside $\triangle ACO$, and line OH intersects side AC.

(b) The range of s/t is the open interval $(4/5, 1)$.

Based on (a), we may consider the configuration shown in the figure. Note that $\angle BOC = 2\angle BAC = 120^\circ$ and

$$\angle BHC = 180^\circ - \angle HBC - \angle HCB = 180^\circ - (90^\circ - \gamma) - (90^\circ - \beta) = 120^\circ,$$



from which it follows that $BOHC$ is cyclic. In particular, $\angle POB = 180^\circ - \angle HOB = \angle HCB = 90^\circ - \beta$, and it follows that

$$\angle APQ = \angle ABO + \angle POB = (90^\circ - \gamma) + (90^\circ - \beta) = 60^\circ.$$

Since $\angle PAQ = 60^\circ$ as well, we see that $\triangle APQ$ is equilateral.

Next note that $\angle POB = 90^\circ - \beta = \angle ACO = \angle QCO$ and $\angle PBO = 90^\circ - \gamma = \angle HBC = \angle HOC = \angle QOC$; since $BO = OC$, we have congruent triangles $\triangle BPO \cong \triangle OQC$. Thus

$$AB + AC = AP + PB + CQ + QA = AP + QO + OP + QA = AP + PQ + QA$$

and so $AP = PQ = QA = (b + c)/3$, where we write $b = AC$ and $c = AB$. Therefore we have

$$\frac{s}{s+t} = \frac{\text{Area}(\triangle APQ)}{\text{Area}(\triangle ABC)} = \frac{AP}{AB}\frac{AQ}{AC} = \frac{((b+c)/3))^2}{bc} = \frac{2 + m + 1/m}{9},$$

where $m = c/b$.

By our assumptions that $b < c$ and $\triangle ABC$ is acute, it follows that the range of m is $1 < m < 2$. (One can see this, for instance, by having A move along the major arc BC from one extreme, where ABC is equilateral and $c/b = 1$, to the other, where $\angle ACB = 90^\circ$ and $c/b = 2$, and noting that c increases and b decreases during this motion.) For $m \in (1, 2)$, the function $f(m) = m + 1/m$ is continuous and increasing: If $1 < m < m' < 2$, then $f(m') - f(m) = (m' - m)(mm' - 1)/(mm') > 0$. Thus the range of $f(m)$ for $m \in (1, 2)$ is $(f(1), f(2)) = (2, 5/2)$. It follows that the range of $s/(s + t) = (2 + f(m))/9$ is $(4/9, 1/2)$, and the range of s/t is $(4/5, 1)$.

This problem and the solution were suggested by Zuming Feng.

3. Same as USAMO 2.
4. Let $f(n) = n + s_b(n)$. For a positive integer m, let $k = \lfloor \log_b(m/2) \rfloor$, so that $m \geq 2b^k$. Note that if $b^m - b^k \leq n < b^m$, then the base b expansion of n begins with $m - k$ digits equal to $b - 1$, and therefore

$$f(n) > b^m - b^k + (m - k)(b - 1) \geq b^m - b^k + (2b^k - k)(b - 1) \geq b^m.$$

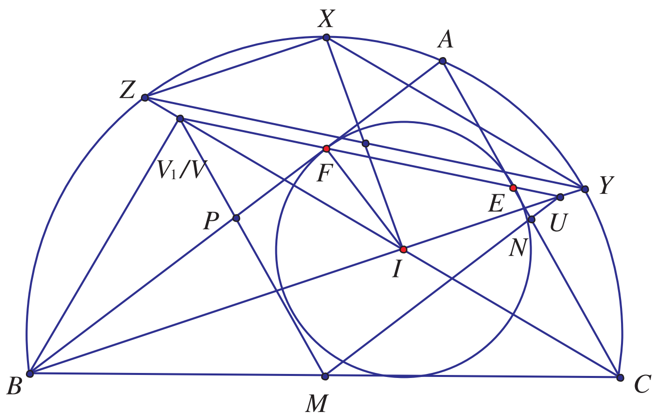
Now consider the set $\{f(1), f(2), \ldots, f(b^m)\}$. Any number that is at most b^m and in the range of f is in this set. However, we see from the above display that $f(n) >$

b^m whenever $b^m - b^k \le n < b^m$. Therefore, there are at least b^k numbers from 1 to b^m that are not in the range of f. Since k goes to infinity as m goes to infinity, the desired result follows.

This problem and the solution were suggested by Palmer Mebane.

5. Same as USAMO 4.

6. Set $\angle ABC = 2y$ and $\angle BCA = 2z$. We first prove the known fact that I lies on ray CV. Let V_1 be the foot of the perpendicular from B to ray CI. Then in right triangle BV_1C, $V_1M = MB = MC$ and $\angle MV_1C = \angle MCV_1 = z = \angle V_1CA$, implying that $\overline{MV_1} \parallel \overline{CA}$; in particular, V_1 lies on line MP. Now $\angle BV_1I = \angle BFI = 90^\circ$, so $BIFV_1$ is cyclic, from which it follows that $\angle V_1FB = \angle V_1IB = y + z = \angle AEF = \angle AFE$; in particular, V_1 lies on $\overline{EF}$. Because V_1 lies on both line MP and line EF, and since $V = V_1$, then V lies on line CI, which is equivalent to the fact that I lies on CV. Likewise we can prove that U lies on line BI.



Rays BI and CI intersect again at Y and Z. Note that $\angle UVC = \angle EVC = \angle AEV - \angle ECV = \angle AEF - \angle ECV = y$. Because $BCYZ$ is cyclic, we have $\angle YZC = \angle YBC = y$. Therefore, $\overline{UV} \parallel \overline{YZ}$. It suffices to show that IX bisects segment $\overline{YZ}$, which is clearly true because $IYXZ$ is a parallelogram. (Indeed, $\angle YZX = \text{arc}(XAY) = \angle XBC - \angle YBC = y + z - y = z = \angle ZYB$, from which it follows that $\overline{ZX} \parallel \overline{IY}$. Likewise, we can show that $\overline{IZ} \parallel \overline{XY}$.)

This problem was suggested by Titu Andreescu and Cosmin Pohoata and the solution was suggested by Zuming Feng.

The top twelve students on the 2014 USAMO were (in alphabetical order):

Joshua Brakensiek	12	Home School (Arizona College Prep–Erie)	AZ
Evan Chen	12	Irvington High School	CA
Ravi Jagadeesan	12	Phillips Exeter Academy	NH
Allen Liu	10	Penfield Senior High School	CT
Nipun Pitimanaaree	12	Pomfret School	NY
Mark Sellke	12	William Henry Harrison High School	IN
Zhuo Qun Song	11	Phillips Exeter Academy	NH
David Stoner	11	South Aiken High School	SC
Kevin Sun	10	Phillips Exeter Academy	NH
James Tao	12	Illinois Mathematics and Science Academy	IL
Alexander Whatley	11	North Houston Academy of Science and Mathematics	TX
Scott Wu	11	Baton Rouge Magnet High School	LA

The top eleven students on the 2014 USAJMO were (in alphabetical order):

Daniel Guo	10	Homestead High School	CA
Samuel Heil	10	Home School (Missouri State University)	MO
Samuel Hsiang	10	Thomas Jefferson High School Science/Tech	VA
Wilbur Li	10	Clements High School	TX
Celine Liang	10	Saratoga High School	CA
James Lin	9	Phillips Exeter Academy	NH
Chad Qian	10	Phillips Exeter Academy	NH
Tiancheng Qin	10	Mission San Jose High School	CA
Alec Sun	9	Phillips Exeter Academy	NH
Felix Wang	9	Roxbury Latin School	MA
Yuan Yao	9	Phillips Exeter Academy	NH

55th International Mathematical Olympiad

PO-SHEN LOH
USA Team Leader
Carnegie Mellon University
Pittsburgh, PA 15213
ploh@cum.edu

Problems

Day 1

1. Let $a_0 < a_1 < a_2 < \cdots$ be an infinite sequence of positive integers. Prove that there exists a unique integer $n \geq 1$ such that

$$a_n < \frac{a_0 + a_1 + a_2 + \cdots + a_n}{n} \leq a_{n+1}.$$

2. Let $n \geq 2$ be an integer. Consider an $n \times n$ chessboard consisting of n^2 unit squares. A configuration of n rooks on this board is *peaceful* if every row and every column contains exactly one rook. Find the greatest positive integer k such that, for each peaceful configuration of n rooks, there is a $k \times k$ square which does not contain a rook on any of its k^2 unit squares.
3. Convex quadrilateral $ABCD$ has $\angle ABC = \angle CDA = 90°$. Point H is the foot of the perpendicular from A to BD. Points S and T lie on sides AB and AD, respectively, such that H lies inside triangle SCT and

$$\angle CHS - \angle CSB = 90°, \quad \angle THC - \angle DTC = 90°.$$

Prove that line BD is tangent to the circumcircle of triangle TSH.

Day 2

4. Points P and Q lie on side BC of acute-angled triangle ABC so that $\angle PAB = \angle BCA$ and $\angle CAQ = \angle ABC$. Points M and N lie on lines AP and AQ, respectively, such that P is the midpoint of AM, and Q is the midpoint of AN. Prove that the lines BM and CN intersect on the circumcircle of triangle ABC.
5. For each positive integer n, the Bank of Cape Town issues coins of denomination $\frac{1}{n}$. Given a finite collection of such coins (of not necessarily different denominations) with total value at most $99 + \frac{1}{2}$, prove that it is possible to split this collection into 100 or fewer groups, such that each group has total value at most 1.
6. A set of lines in the plane is in *general position* if no two are parallel and no three pass through the same point. A set of lines in general position cuts the plane into regions, some of which have finite area; we call these its *finite regions*. Prove that for all sufficiently large n, in any set of n lines in general position it is possible to color at least $\sqrt{n}$ of the lines blue in such a way that none of its finite regions has a completely blue boundary.

   Note: Results with $\sqrt{n}$ replaced by $c\sqrt{n}$ will be awarded points depending on the value of the constant c.

Math. Mag. **87** (2014) 310–317. doi:10.4169/math.mag.87.4.310.

Solutions

In honor of our students' excellent performance, we have invited each of them to submit one solution along with their own perspective.

1 (Solution by Mark Sellke) The desired condition rearranges to the following pair of inequalities:

$$\sum_{i=0}^{n-1} a_i > (n-1)a_n \quad \text{and} \quad \sum_{i=0}^{n} a_i \leq n a_{n+1}.$$

Noting the similarity between the two, we define S_k (for $k \geq 0$) as

$$S_k = (k-1)a_k - \sum_{i=0}^{k-1} a_i.$$

Reinterpreting the condition in terms of this new sequence, we wish to show there is a unique n such that $S_n < 0 \leq S_{n+1}$. To do this, we show that the sequence S_k is strictly increasing after S_1 and that $S_1 < 0$. As S_k is a sequence of integers, this will imply the desired conclusion; take S_n to be the last negative value in the sequence.

First, $S_0 = S_1 = -a_0 < 0$, as the a_i are given to be positive. Next, note that when $k \geq 1$,

$$S_{k+1} - S_k = k a_{k+1} - \sum_{i=0}^{k} a_i - (k-1)a_k + \sum_{i=0}^{k-1} a_i = k(a_{k+1} - a_k) > 0.$$

Hence, S_k is a strictly increasing sequence of integers, starting from $S_1 < 0$. So there is a unique n with $S_n < 0 \leq S_{n+1}$ and we are done. ■

This problem was proposed by Austria.

2 (Solution by Joshua Brakensiek) Let C be the chessboard and let $f(n) = \lfloor\sqrt{n-1}\rfloor$. We claim that the greatest possible value of k is $f(n)$. The proof is divided into two parts.

Part 1. Every peaceful configuration of n rooks on C contains an empty $f(n) \times f(n)$ square.

Proof. Consider an $f(n) \times n$ rectangle C' of C which contains the unique rook in the first column. There are $n - f(n) + 1$ squares of C' of size $f(n) \times f(n)$. Since any of the $f(n)$ rooks in C' can overlap with at most $f(n)$ of these squares, and the rook in the first column overlaps with only 1 of these squares, there are at most $f(n)^2 - f(n) + 1$ squares of C' which contain a rook. Since n is strictly greater than $f(n)^2 = \lfloor\sqrt{n-1}\rfloor^2$, we have that $n - f(n) + 1 > f(n)^2 - f(n) + 1$. Hence, there exists an $f(n) \times f(n)$ square that does not contain a rook.

Part 2. There exists a peaceful configuration of n rooks that contains no empty $(f(n)+1) \times (f(n)+1)$ square.

Proof. We divide this part into two cases according to whether n is a perfect square. To describe the locations of rooks, we number the rows and columns from 0 to $n-1$ and give each rook coordinates (r, c), where r is its row and c its column.

Case 1. If n is a perfect square, let $d = \sqrt{n}$. We place the rooks at $(i + dj, j + di)$, where $i, j \in \{0, \ldots, d-1\}$. This configuration is peaceful because $i + dj$ and $j + di$

are both bijective mappings from $\{0, \ldots, d-1\}^2$ to $\{0, \ldots, n-1\}$. Since $f(n) = d-1$, we need to show that this configuration has no empty $d \times d$ square. Assume, for sake of contradiction, that there exists an empty $d \times d$ square occupying rows a through $a+d-1$. If we let $a = i_0 + dj_0$, then the rooks on rows $a, \ldots, a+d-1$ occupy columns

$$j_0 + di_0,\ j_0 + d(i_0+1), \ldots, j_0 + d(n-1),\ j_0 + 1 + d(0), \ldots, j_0 + 1 + d(i_0 - 1),$$

respectively. This list when sorted is

$$j_0 + 1, \ldots, j_0 + 1 + d(i_0 - 1),\ j_0 + di_0, \ldots, j_0 + d(n-1).$$

Notice that every pair of consecutive columns are either d or $d-1$ apart and that the first and last columns are within d columns of the nearest side of the grid. Thus, it is impossible for an empty $d \times d$ square to fit in rows a through $a+d-1$, a contradiction. Therefore, the stated configuration has no empty $(f(n)+1) \times (f(n)+1)$ square.

Case 2. If n is not a perfect square, let n' be the least perfect square greater than n. Consider a peaceful configuration of n' rooks on an $n' \times n'$ chessboard which contains no empty $(f(n')+1) \times (f(n')+1)$ square. Remove the first $n' - n$ rows of this chessboard to form an $n \times n'$ chessboard. Since $f(n) = f(n')$, this new chessboard has no empty $(f(n)+1) \times (f(n)+1)$ square. Now, remove the $n' - n$ columns of this chessboard which are empty, and compact the remaining ones into an $n \times n$ chessboard. This $n \times n$ chessboard contains a peaceful configuration of rooks with no empty $(f(n)+1) \times (f(n)+1)$ square.

Thus, the maximal value of k is $f(n)$, as desired. ■

The author would like to thank Mark Sellke for a cleaner approach to Case 2.

This problem was proposed by Croatia.

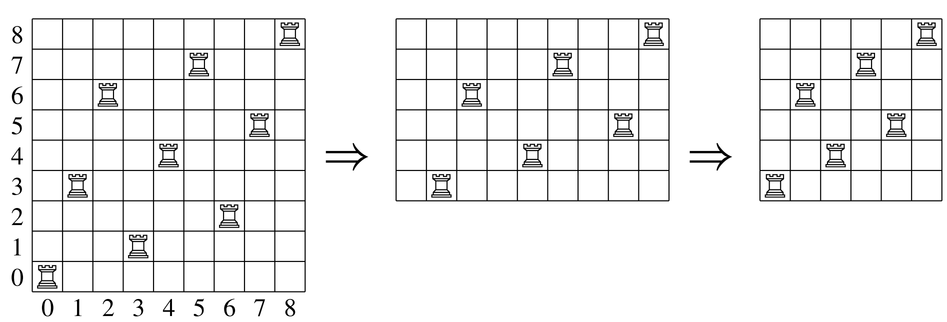


Figure 1 A peaceful configuration of 9 rooks without an empty 3×3 square can be reduced to a peaceful configuration of 6 rooks, still without an empty 3×3 square. Notice that $f(9) + 1 = f(6) + 1 = 3$.

3 (Solution by Yang Liu) This solution demonstrates the power of *inversion* and *symmedians*. Consider the diagram in FIGURE 2. Since $\angle ABC + \angle CDA = 180°$, quadrilateral $ABCD$ is cyclic. Define the points C_B, C_D to be the reflections of C over B, D respectively. Now, we can show two simple facts about these points.

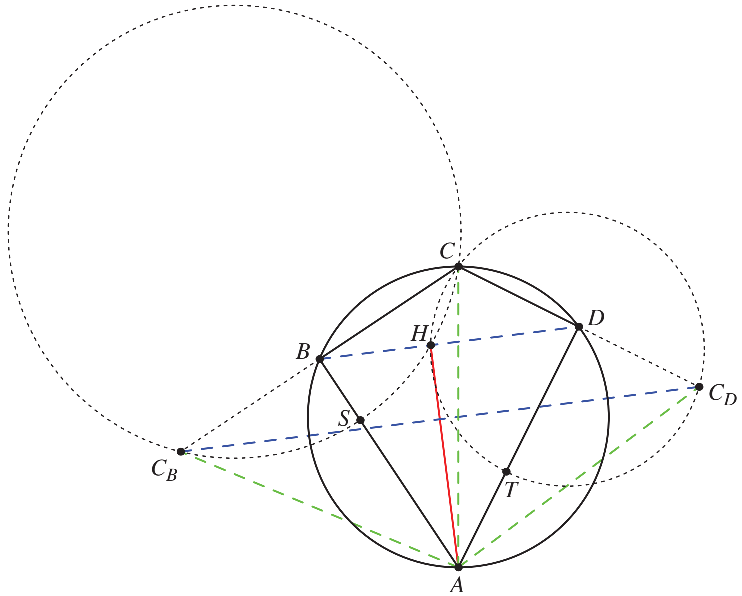


Figure 2 The given configuration, before inversion

LEMMA 1. $HC_B = HC_D$.

Proof. Since B, D are the midpoints of CC_B, CC_D, respectively, $BD \parallel C_BC_D$. Also, since $AH \perp BD$, we have that $AH \perp C_BC_D$. But also note that since $\angle ABC = \angle CDA = 90^\circ$, $AC_B = AC = AC_D$. Since $AH \perp C_BC_D$, AH must be the perpendicular bisector of C_BC_D, hence $HC_B = HC_D$ also. ■

LEMMA 2. *Quadrilaterals $CHSC_B$ and $CHTC_D$ are cyclic.*

Proof. Since C_B is the reflection of C over B, we know that $CS = C_BS$, and therefore, $\angle SC_BC = \angle SCC_B = 90^\circ - \angle CSB = 180^\circ - \angle CHS$ by the condition that $\angle CHS - \angle CSB = 90^\circ$. Therefore, $\angle CHS + \angle SC_BC = 180^\circ$, which implies that $CHSC_B$ is cyclic. The cyclicity of $CHTC_D$ follows similarly. ■

Now we actually have three circles and a line passing through the point H: the circumcircles of $CHSC_B$, $CHTC_D$, HST, and line BD. This motivates us to perform an inversion centered at H with arbitrary radius. An inversion centered at H with radius r maps each point P in the plane to the point P' on the ray HP such that $HP \cdot HP' = r^2$. A well-known property of an inversion is that for any points P, Q in the plane, $\triangle HPQ \sim \triangle HQ'P'$. Indeed, both triangles share the angle $\angle PHQ = \angle P'HQ'$ and

$$\frac{HP}{HQ} = \frac{r^2/HP'}{r^2/HQ'} = \frac{HQ'}{HP'}.$$

This similarity implies that $\angle HPQ = \angle HQ'P'$ for any points P, Q.

Inversion is a conformal map; that is, it preserves angles locally. Inversion also maps circles to other circles, where lines are circles of infinite radius. More specifically, circles through H map to lines not through H, and vice versa. Also, lines through H map to themselves, and circles not through H map to circles not through H. After inverting at H, our diagram becomes FIGURE 3, where the notation P' is used to indicate the image of P, and H is left in the diagram.

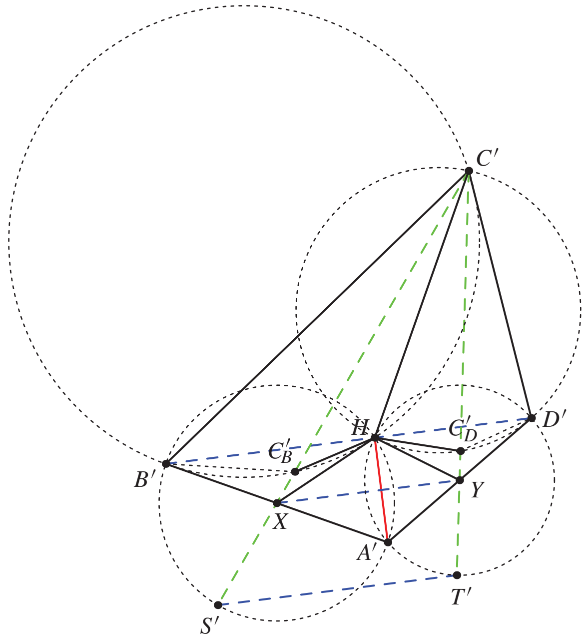


Figure 3 After inversion

Most importantly, in order to prove that circle HST is tangent to BD, we must now show that $S'T' \parallel B'D'$. This is because the circle HST maps to line $S'T'$, line BD maps to line $B'D'$, and inversion preserves angles between curves.

Because lines not through H map to circles through H, $C'HC'_BB'$ is a cyclic quadrilateral (C, B, C_B were collinear). Also, since $CHSC_B$ is cyclic, now C'_B, C', S' are collinear. Finally, since A, B, S are collinear, $HA'S'B'$ is cyclic. Define X to be the midpoint of $A'B'$. Similarly define Y to be the midpoint of $A'D'$. Since $\angle A'HB' = 90^\circ$, X is the center of $HA'S'B'$. Also, from our discussion of inversion,

$$\angle XB'H = \angle A'B'H = 90^\circ - \angle HA'B' = 90^\circ - \angle HBA = \angle HBC = \angle HC'B'.$$

Therefore, XB' is tangent to circle $C'HC'_BB'$. But since $XH = XB'$, XH is also a tangent to the circle. Therefore, X is the intersection of the tangents to circle $C'HC'_BB'$ at B' and H.

Before continuing, we note some important properties of a symmedian in a triangle UVW through U. This is a line through U that is *isogonal* to the median through U, meaning the reflection of the median through U across the angle bisector of $\angle WUV$. Define X to be the intersection of the tangents to circle UVW at V and W, and let $Z \neq U$ be a point on circle UVW satisfying $UV/UW = ZV/ZW$. Then U, Z, X are collinear, and they determine the symmedian through U.

Remember from the above discussion of inversion that $\triangle HCC_B \sim \triangle HC'_BC'$. Note that HB was a median in $\triangle HCC_B$. Since the line HB' is the same as the line HB, line HB' is isogonal to the median of $\triangle HC'_BC'$, so HB' is a symmedian. From the above discussion, $B'C'/B'C'_B = HC'/HC'_B$, which rearranges to $C'B'/C'H = C'_BB'/C'_BH$. Therefore, $C'C'_B$ is a symmedian in $\triangle C'B'H$. Since X is the intersection of the tangents at B' and H, we find that C', C'_B, X are collinear. But C', C'_B, S' were collinear, so C', X, S' are collinear. Similarly, C', Y, T' are collinear.

Since X, Y are the midpoints of $A'B'$, $A'D'$ respectively, $XY \parallel B'D'$. Therefore, it suffices to show that $XY \parallel S'T'$. This is now equivalent to showing the ratio equality $XC'/XS' = YC'/YT'$. Since X is the center of circle $B'HA'S'$, $XS' = XH$. So $C'X/XS' = C'X/XH$. But now a simple angle chase yields that $\triangle XHC'_B \sim \triangle XC'H$. Therefore, $XC'/XH = C'H/HC'_B$. But $HC'_B = HC'_D$ by Lemma 1, so

$$\frac{C'H}{HC'_B} = \frac{C'H}{HC'_D} = \frac{YC'}{YH} = \frac{YC'}{YT'},$$

finishing the proof. ■

This problem was proposed by Iran.

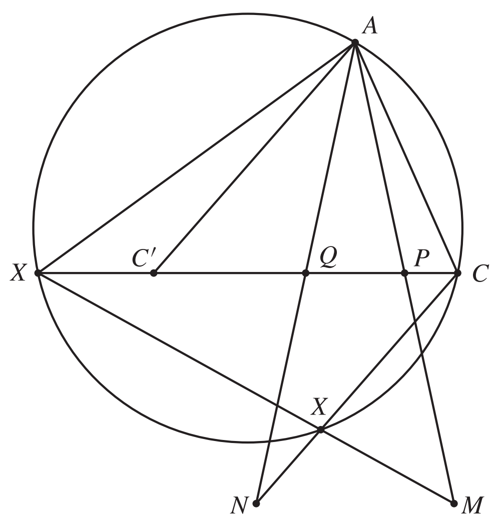


Figure 4 Problem 4

4 (Solution by Allen Liu) Let BM and CN intersect at X, and let C' be the reflection of C over Q, as in FIGURE 4. Note that $\triangle CBA$ is similar to $\triangle CAQ$ since we are given that $\angle CAQ = \angle ABC$. Similarly, we get that $\triangle BAP$ is similar to $\triangle BCA$. Hence, $\triangle BAP$ is similar to $\triangle ACQ$.

Now C' is the reflection of C over Q while M is the reflection of A over P. Hence, we have that $\triangle BPM$ and $\triangle AQC'$ are similar. We get that $\angle CBX = \angle PBM = \angle QAC'$. Also, by the definition of C', $ACNC'$ is a parallelogram. Hence $\angle XCB = \angle CC'A$.

Now

$$\begin{aligned}\angle BXC &= 180 - \angle XBC - \angle XCB = 180 - \angle QAC' - \angle CC'A \\ &= \angle AQC' = 180 - \angle AQC = 180 - \angle BAC.\end{aligned}$$

(We know that $\angle AQC = \angle BAC$ because $\triangle CBA \sim \triangle CAQ$.)

Hence we have that quadrilateral $ACXB$ is cyclic; that is, X is on the circumcircle of ABC, as desired. ■

This problem was proposed by Georgia.

5 (Solution by James Tao) This problem states that any finite set of unit fractions with total size at most $99 + 1/2$ can be partitioned into 100 groups each of size at most 1. This solution replaces 100 with n and $99 + 1/2$ with $n - 1/2$. We view the unit fractions as blocks to be put in boxes of size 1.

The solution involves two ideas that are applicable to all such packing problems (such as problem C1 on the 2013 shortlist): merging and the greedy algorithm. Merging says that packing becomes strictly harder when two blocks are "merged," i.e., constrained to lie in the same box. The greedy algorithm says that, unless every box has less empty space than the size of the next unpacked block, we can continue packing.

Suppose for sake of contradiction that a set of unit fractions exists that cannot be packed. Among these counterexamples, pick the one with smallest n and, among those, the smallest number of blocks. Packing is trivial if $n = 1$, so $n \geq 2$. If two blocks of size $1/(2k)$ or $2k + 1$ blocks of size $1/(2k + 1)$ exist, they may be merged into one block of size $1/k$ or one block of size 1, respectively. When $k \geq 1$, this decreases the number of blocks and produces a smaller counterexample, a contradiction. If any block has size 1, pack it in its own box; the remaining blocks of total size $n - 1 - 1/2$ can be packed into $n - 1$ boxes, again a contradiction. Therefore there are at most $2k$ blocks of size $1/(2k + 1)$ and at most one of size $1/(2k)$, for all $k \geq 1$, and no blocks of size 1.

Because

$$\frac{2(k-1)}{2(k-1)+1} + \frac{1}{2k} < \frac{2(k-1)}{2(k-1)+1} + \frac{1}{2(k-1)+1} = 1,$$

we may pack all blocks of size $1/(2(k - 1) + 1)$ and $1/(2k)$ into box k, for $1 \leq k \leq n$. All remaining blocks have size at most $1/(2n + 1)$, and we attempt to pack them using the greedy algorithm. This can fail only if the space in each box is less than the size of some unpacked block. Since the blocks have total size $n - 1/2$ but the boxes have total capacity n, the total amount of space is at least $1/2$, hence the space in some box is at least $1/(2n)$. So failure of the greedy algorithm implies that $1/(2n) < 1/(2n + 1)$, our final contradiction. This shows that no counterexamples exist. ∎

This problem was proposed by Luxembourg.

6 (Solution by Sammy Luo) For simplicity, assume $n \geq 3$, so that finite regions exist.

Let k be the maximum number of lines that can be colored blue without creating any finite regions with a completely blue boundary. Consider such a set A of k lines and color them blue. Since k is maximal, this set cannot be extended. Let $\overline{A}$ be the set of lines not in A, so for every line $\ell \in \overline{A}$, there exists a finite region with ℓ as a side such that all other sides are in A. Call such a finite region *ℓ-dependent*.

Let I be the set of intersections of distinct pairs of lines in A. We associate each line $\ell \in \overline{A}$ to an element of I as follows: For each ℓ, consider the set of vertices of ℓ-dependent regions that do not lie on ℓ. Associate ℓ to the vertex in this set that is closest to ℓ. So each element of $\overline{A}$ is associated to an element of I. We claim that in this association, at most two lines ℓ are associated to each element of I.

Consider an intersection $P \in I$. Note that the two lines a, b meeting at P form four "angles," each of which belongs to at most one finite region and so can be used in at most one association. We claim that no two adjacent angles can be used. Assume otherwise, so there are lines $\ell, \ell' \in \overline{A}$ forming dependent regions with adjacent angles at P. Without loss of generality, the configuration is as in FIGURE 5.

Here, the angle α at P is contained in an ℓ-dependent region, and the angle β at P is contained in an ℓ'-dependent region. Lines ℓ and ℓ' intersect the blue ray from P shared by these regions at R and T, respectively, and without loss of generality $PR < PT$.

If the ℓ'-dependent region containing β had a vertex X on the interior of segment PT, ℓ' would be closer to X than to P, and so would not be associated to P. So since

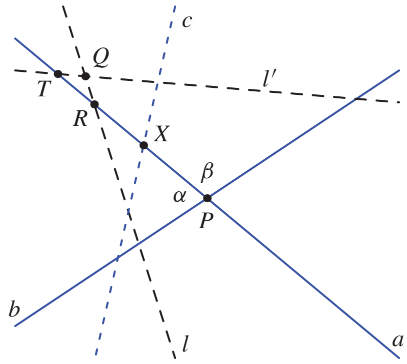


Figure 5 Configuration at P

the boundary of this region contains some portion of ray PT, it must contain vertex T. But line ℓ intersects segment PT at R, so P, T cannot be contained in the same region, a contradiction.

So, in fact, no two adjacent angles at P can be used to associate lines in $\overline{A}$ to P, and so at most 2 elements of $\overline{A}$ can be associated to each element P of I. Thus

$$|\overline{A}| \leq 2|I|.$$

But $|I| = \binom{k}{2}$ and $|\overline{A}| = n - |A| = n - k$, so $n - k \leq 2\binom{k}{2} = k^2 - k$, which implies $k \geq \sqrt{n}$ as desired. ■

This problem was proposed by Austria.

Results

The IMO was held in Cape Town, South Africa, on July 8–9, 2014. There were 560 competitors from 101 countries and regions. On each day contestants were given four and a half hours for three problems.

The top score of 42/42 was shared by Jiyang Gao (China), Alexander Gunning (Australia), and Po-Sheng Wu (Taiwan). The USA team won 5 gold and 1 silver medals, placing second behind China. The students' individual results were as follows.

- Joshua Brakensiek, who finished 12th grade (homeschooled) in Phoenix, AZ, won a silver medal.
- Allen Liu, who finished 10th grade at Penfield Senior High School in Penfield, NY, won a gold medal.
- Yang Liu, who finished 11th grade at Ladue Horton Watkins High School in St. Louis, MO, won a gold medal.
- Sammy Luo, who finished 12th grade at North Carolina School of Science and Math in Durham, NC, won a gold medal.
- Mark Sellke, who finished 12th grade at William Henry Harrison High School in West Lafayette, IN, won a gold medal.
- James Tao, who finished 12th grade at Illinois Mathematics and Science Academy in Aurora, IL, won a gold medal.

2014 Carl B. Allendoerfer Awards

The Carl B. Allendoerfer Awards, established in 1976, are made to authors of articles of expository excellence published in *Mathematics Magazine*. The Awards are named for Carl B. Allendoerfer, a distinguished mathematician at the University of Washington and President of the Mathematical Association of America, 1959–60.

Sally Cockburn and Joshua Lesperance, "Deranged Socks," *Mathematics Magazine,* **86:2** (2013), pages 97–109.

How many ways can n people each choose two gloves from a pile of n distinct pairs of gloves, so that nobody gets a matching pair? In this article, authors Sally Cockburn and Joshua Lesperance consider a fun and surprisingly challenging twist on this familiar combinatorics problem, replacing gloves with socks. They ask, "How many ways can n people each choose two socks from a pile of n distinct pairs of socks, with no one getting a matching pair?".

The sock problem extends the glove problem by removing the crucial assumption that right- and left-handed gloves are distinguishable. In the glove problem, the matched pairs are effectively sorted in two piles: left-handed gloves in one and right-handed gloves in another, and since each person will take one left-handed and one right-handed glove, the ways to derange gloves are limited. But we do not distinguish between left and right socks, and this allows for a wider range of possibilities. As the authors illustrate, the seemingly innocuous switch from gloves to socks significantly complicates the problem.

The authors begin their discussion with counting derangements, permutations in which every object gets moved. From there, they develop solutions to the more complicated sock problem, starting with a recursive formula. Next they come up with a non-recursive solution, and as the authors develop their ideas, the reader becomes thoroughly engaged by the connections of this problem with other mathematical results. The reader is led on a lively tour of a variety of discrete mathematical tools: partitions, cyclic permutations, recurrence relations, ordinary and exponential generating functions. At the end the authors deliver a final pleasing touch: using complex analysis to show that the fraction of all sock distributions that are deranged in this sense converges to $1/e$.

Response from Sally Cockburn and Joshua Lesperance We were honored and delighted to learn that our paper had received a Carl B. Allendoerfer Award. Truth to tell, we were honored and delighted when it was accepted for publication in such a popular and respected journal as *Mathematics Magazine*. The MAA makes an immeasurable contribution to the field through their publication of such journals as *The American Mathematical Monthly*, *Mathematics Magazine*, and *The College Mathematics Journal*. Special thanks also to Walter Stromquist, for his insightful and encouraging feedback throughout the publication process. Finally, we are immensely grateful to the student who first innocently suggested in class the sock variation on the glove problem. At first, we blithely assigned it as an extra credit homework assignment, but then, after nobody (including us) managed to solve it by the next class, or indeed, by the end of the semester, we realized that we had happened upon a gem of a problem.

Math. Mag. **87** (2014) 318–320. doi:10.4169/math.mag.87.4.318. © Mathematical Association of America

Biographical Notes

Sally Cockburn was born and raised in Ottawa, Canada. She first fell in love with mathematics at Queen's University, where she completed a Bachelor of Science in 1982 and an Master's of Science in 1984. While at Yale University pursuing a Ph.D., her research in algebraic topology took an unexpected detour into generating functions and combinatorial identities, and she became hooked on discrete mathematics. Since joining the Mathematics Department at Hamilton College in 1991, her teaching and research specialization has been in combinatorics, graph theory and linear optimization, although she also likes to dabble in the philosophy of mathematics. Sally is an avid squash player and helped coach Hamilton's varsity squash teams for ten years. Whenever possible, she escapes to go hiking, biking, kayaking and skiing in the wilds of the Adirondacks.

Joshua Lesperance has a B.S. in applied mathematics from Rochester Institute of Technology and both a M.S. and a Ph.D. from the University of Notre Dame, where he studied algebraic geometry. He has taught mathematics at Hamilton, Oberlin, Skidmore, and Franklin & Marshall colleges. Since leaving Notre Dame, his research interests have shifted back towards his applied mathematics roots, most recently working on applications of spherical harmonics in the understanding of human perception of 3-dimensional shape. Joshua currently lives in Columbus, Ohio with his wife Karilyn and their two Siberian Huskies, Mia and Kai.

Susan Marshall and Donald Smith, "Feedback, Control, and Distribution of Prime Numbers," *Mathematics Magazine,* **86:3** (2013), pages 189–203.

In this article, Susan Marshall and Donald Smith describe an unusual application of a technique of mathematical modeling, feedback and control, to a classical mystery of number theory, the distribution of primes. In a famous result due to Gauss, the density of primes is (approximately) inversely proportional to the natural logarithm. The differential equation below reasonably models the density of primes. Here $f(x)$ represents the density of primes:

$$f'(x) = \frac{f(x)f(\sqrt{x})}{2x}$$

Although this is a known application in differential equation literature, it appears to be largely forgotten in number theory. In the process of deriving this model, the authors give the reader a lively introduction to the theory of feedback and control, complete with a cast of characters representing different feedback phenomena in the face of perturbations. We have the "cool, calm, and collected" responder, the "whimsical" responder, and finally the "panicky and overreacting" responder. The authors note that the distribution of prime numbers has an element of randomness, yet it also stays on track, much like a feedback and control system, with either a "whimsical" or a "cool, calm, and collected" response.

The authors demonstrate how one verifies not only that the differential equation (above) predicts the correct density function, but also that the model is robust. That is, while the true density of primes at times deviates from $1/\ln x$, they show that $1/\ln x$ is the ideal path of the true density. For a "perturbed solution" to the differential equation, as x increases we see that $f(x)$ approaches $1/\ln x$. This represents stability. The mathematics is presented as a beautifully simple (manageable) change of variables. With stability comes the conclusion that the model predicts the prime number theorem. After further computation, the authors show that Littlewood's Theorem is also predicted by the model. For a complete lesson in modeling, the authors also describe the limitations of their model; it is successful as far as gross behavior goes, but most likely fails at the fine scale, as it is incompatible with the Riemann Hypothesis. In this engrossing article, descriptions and arguments are interspersed with history, which serves to round out a satisfying tour through both prime density and mathematical modeling.

Response from Susan Marshall and Donald Smith We are excited and grateful for the recognition of our work with a Carl B. Allendoerfer Award! Our collaboration (between a professor of Mathematics and a professor of Business) began on a Search Committee on which we both served. Upon learning that Susan's area of research is number theory, Bob (whose training is in Operations Research) shared his own passion for the subject and mentioned a differential equation he had discovered that seemed to model the density of prime numbers. The allure of this equation is that it captures the intuitive idea of self-regulation that appears to be at the heart of the distribution of prime numbers—namely that if there are "too many" primes in an interval, there will be fewer subsequent primes (and vice-versa). This gives the system of prime numbers the appearance of a feedback and control system. After much digging around, we discovered that Bob was not the first to discover the equation and was in the company of no less than Lord Cherwell, scientific advisor to Churchill during WWII. Given the interesting history, the interplay of different fields of mathematics, and the accessibility of the topic, we felt *Mathematics Magazine* was an appropriate venue to share our story. We are indebted to Editor Walter Stromquist for championing the project through many, many revisions. Thanks also to our colleagues and our families for their support throughout the process.

Biographical Notes

Susan H. Marshall received a BS in Mathematics from Wake Forest University in 1993, with a minor in Psychology. After a brief stint as a data analyst for the Hubble Space Telescope at Goddard Space Flight Center in Maryland, she returned to school and received a PhD in Mathematics from the University of Arizona in 2001. While in graduate school, Susan studied Arithmetic Geometry. She was a postdoctoral fellow at the University of Texas at Austin from 2001–2004. She is currently an Associate Professor of Mathematics at Monmouth University, where she has just completed her 10th year. She lives on the Jersey Shore with her husband (and colleague) David, and their two children Gillian and Dylan.

Donald R. (Bob) Smith received an AB in Physics (magna cum laude) from Cornell in 1969, a MS in Operations Research from Columbia University in 1974 and a PhD in Operations Research from the University of California at Berkeley in 1975. He was an Assistant Professor of Operations Research at Columbia University from 1975–1979, before working at Bell Laboratories as a Member of Technical Staff and a Supervisor from 1980–2001. After leaving Bell Laboratories, he joined the faculty at Monmouth University where he is currently an Associate Professor in the Management and Decision Science Department. Most of his publications are in Operations Research journals in the areas of stochastic processes. Bob has always been fascinated by prime numbers because they are a deterministic system with elements of apparent randomness but hidden control. He and his wife Pat have 3 grown children and two grandchildren. He is an avid cyclist averaging over 11K miles per year.

UNITED STATES POSTAL SERVICE®

Statement of Ownership, Management, and Circulation
(All Periodicals Publications Except Requester Publications)

1. Publication Title Mathematics Magazine	2. Publication Number 0 0 2 5 _ 5 7 0 X	3. Filing Date September 5, 2014
4. Issue Frequency Bimonthly except July and August	5. Number of Issues Published Annually 5	6. Annual Subscription Price $264
7. Complete Mailing Address of Known Office of Publication *(Not printer) (Street, city, county, state, and ZIP+4®)* Mathematical Association of America 1529 Eighteenth St., NW Washington, DC 20036		Contact Person Beverly Ruedi Telephone *(Include area code)* 540-582-2208 (home office)

8. Complete Mailing Address of Headquarters or General Business Office of Publisher *(Not printer)*
Mathematical Association of America
1529 Eighteenth St., NW
Washington, DC 20036

9. Full Names and Complete Mailing Addresses of Publisher, Editor, and Managing Editor *(Do not leave blank)*

Publisher *(Name and complete mailing address)*
Mathematical Association of America
1529 Eighteenth St., NW
Washington, DC 20036

Editor *(Name and complete mailing address)*
Walter Stromquist
132 Bodine Road
Berwyn, PA 19312-1027

Managing Editor *(Name and complete mailing address)*
Beverly Ruedi
1529 Eighteenth Street, NW
Washington, DC 20036

10. Owner *(Do not leave blank. If the publication is owned by a corporation, give the name and address of the corporation immediately followed by the names and addresses of all stockholders owning or holding 1 percent or more of the total amount of stock. If not owned by a corporation, give the names and addresses of the individual owners. If owned by a partnership or other unincorporated firm, give its name and address as well as those of each individual owner. If the publication is published by a nonprofit organization, give its name and address.)*

Full Name	Complete Mailing Address
Mathematical Association of America	1529 18th St., NW, Washington, DC 20036

11. Known Bondholders, Mortgagees, and Other Security Holders Owning or Holding 1 Percent or More of Total Amount of Bonds, Mortgages, or Other Securities. If none, check box → ☒ None

Full Name	Complete Mailing Address

12. Tax Status *(For completion by nonprofit organizations authorized to mail at nonprofit rates) (Check one)*
The purpose, function, and nonprofit status of this organization and the exempt status for federal income tax purposes:
☒ Has Not Changed During Preceding 12 Months
☐ Has Changed During Preceding 12 Months *(Publisher must submit explanation of change with this statement)*

PS Form **3526,** September 2007 *(Page 1 of 3 (Instructions Page 3))* PSN 7530-01-000-9931 **PRIVACY NOTICE**: See our privacy policy on *www.usps.com*

13. Publication Title	14. Issue Date for Circulation Data Below	
Mathematics Magazine	September 2014	
15. Extent and Nature of Circulation	**Average No. Copies Each Issue During Preceding 12 Months**	**No. Copies of Single Issue Published Nearest to Filing Date**
a. Total Number of Copies *(Net press run)*	3000	3100
b. Paid Circulation *(By Mail and Outside the Mail)* (1) Mailed Outside-County Paid Subscriptions Stated on PS Form 3541(Include paid distribution above nominal rate, advertiser's proof copies, and exchange copies)	2700	2800
(2) Mailed In-County Paid Subscriptions Stated on PS Form 3541 *(Include paid distribution above nominal rate, advertiser's proof copies, and exchange copies)*	0	0
(3) Paid Distribution Outside the Mails Including Sales Through Dealers and Carriers, Street Vendors, Counter Sales, and Other Paid Distribution Outside USPS®	0	0
(4) Paid Distribution by Other Classes of Mail Through the USPS (e.g. First-Class Mail®)	0	0
c. Total Paid Distribution *(Sum of 15b (1), (2), (3), and (4))*	2700	2800
d. Free or Nominal Rate Distribution *(By Mail and Outside the Mail)* (1) Free or Nominal Rate Outside-County Copies included on PS Form 3541	0	0
(2) Free or Nominal Rate In-County Copies Included on PS Form 3541	0	0
(3) Free or Nominal Rate Copies Mailed at Other Classes Through the USPS (e.g. First-Class Mail)	0	0
(4) Free or Nominal Rate Distribution Outside the Mail *(Carriers or other means)*	100	100
e. Total Free or Nominal Rate Distribution *(Sum of 15d (1), (2), (3) and (4))*	100	100
f. Total Distribution *(Sum of 15c and 15e)*	2800	2900
g. Copies not Distributed *(See Instructions to Publishers #4 (page #3))*	200	200
h. Total *(Sum of 15f and g)*	3000	3100
i. Percent Paid *(15c divided by 15f times 100)*	96%	96%

16. Publication of Statement of Ownership
☒ If the publication is a general publication, publication of this statement is required. Will be printed in the December 2014 issue of this publication. ☐ Publication not required.

17. Signature and Title of Editor, Publisher, Business Manager, or Owner
Beverly Joy Ruedi
Managing Editor
Date: 9/5/2014

I certify that all information furnished on this form is true and complete. I understand that anyone who furnishes false or misleading information on this form or who omits material or information requested on the form may be subject to criminal sanctions (including fines and imprisonment) and/or civil sanctions (including civil penalties).

PS Form **3526,** September 2007 *(Page 2 of 3)*

MATHEMATICAL ASSOCIATION OF AMERICA
1529 Eighteenth St., NW
Washington, DC 20036

CONTENTS